TOWARD A KNOWLEDGE-BASED EXPERT SYSTEM OF CAPITAL BUDGETING DECISION MAKING

by

JAN-SUAN LIN

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Approved by:

Ching-Lai Hwang

Major Professor

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Chap 1. INTRODUCTION

1.1 Capital Budgeting

Osteryoung [31] classified the firm's financial decisions into two categories: capital structure and capital investment. Capital structure decisions are related to the amounts and sources of funds available to the firm, for example, the issuance of new stocks or borrowing money from lenders. Capital investment decisions concern the use of funds from whatever source. The definition of capital investment offered by Osteryoung [31] is as follows :

> "A capital investment may be defined as one which requires a current outlays or series of future outlays of cash resources in return for an anticipated flow of future benefits."

In consideration of time dimension, the capital investment can be classified into two sub-categories: working capital decisions and capital budgeting decisions. Working capital decisions refer to short-term (within one fiscal period) policies on the use of funds, for example, the management of cash. Capital budging decisions refer to the long-term (more than one fiscal period) investments, such as the investment on land, building, equipment and facilities.

Clark [5] further defines capital budgeting as follows :

> "Capital budgeting is the decision area in financial management which establishes goals and criteria for investing resources in long-term projects."

According to Clark, the components of capital budgeting analysis include the forecast of the benefit/cost of the project, discounting the funds invested in the project at an appropriate rate, assessing the risk associated with the project and following up to determine if the project is performing as expected.

1.2 Expert System

An expert system is a computer program that solves problems in much the same manner as human experts. Feigenbaum [10] stated that:

> "An expert system is an intelligent computer program that uses knowledge and inference procedures to solve problems that are difficult enough to require significant human expertise for their solution."

Knowledge and the inference procedures can be thought of as a model of the expertise which may act as an expert in the expert system. Feigenbaum called those who build knowledge-based expert systems "knowledge engineer."

The basic structure of an expert system is shown in Figure 1.1.

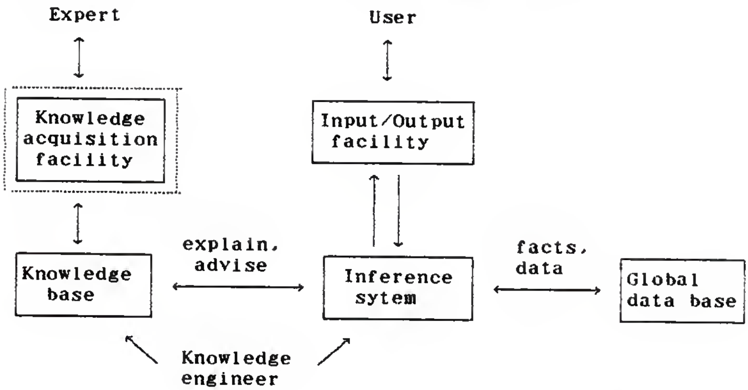


Figure 1.1 Basic structure of an expert system

(Source: Feigenbaum [10] revised by Hwang [17])

There are five components in an expert system: (1) The input/output system which allows the user to communicate with the system; (2) The global data base which stores the input data for the specific problem entered by the user; (3) A knowledge base which contains the basic knowledge of the domain expertise; (4) An inference system that incorporates reasoning methods which acts upon the global data base and the knowledge base to solve the stated problem; (5) (Perhaps) a knowledge acquisition facility which allows the system to acquire further knowledge from experts.

1.3 Purpose of The Study

The current usage of capital budgeting techniques has been surveyed by Liberator [26] in 1983 on R&D project management sampling from "Fortune 500" firms and Khan [22] in 1987 on capital budgeting practices sampling from U. S. large cities. **Their surveys revealed (see Table 1.1 and 1.2) that** the usage of capital budgeting techniques is still limited.

Table 1.1 Technique Usage by Category [26]

Technique Category	Number of Respondents Using: no tech.	1 tech.	2 or more tech.
Financial	8	7	25
Risk Assesment	26	11	3
Budgeting	27	13	N.A.
Scheduling	9	10	21
Math. Progrm.	40	0	0
Behavioral	40	0	0
Subjective	25	15	0

Table 1.2 Capital Budgeting Technique Usage [22]

1. Technique Nature	Response [No]	Response [%]
Informal	44	44.9
Formal	2	2.0
Formal-Informal	52	53.1
Total	98	100%

2. Analytical Techniques	Response [No]	Response [%]
NPV	17	31.5
IRR	4	7.4
B-C ratio	32	59.5
Total:	54	100%

3. New Techniques	Response [No]	Response [%]
None	97	90.7
Decision Theory	2	1.9
Math. Programming	4	3.7
CPM-PERT	4	3.7
Total	107	100%

Khan concluded that the majority of respondents lacked knowiedge of modern techniques. They preferred simpie, informal methods to more sophisticated techniques, and they considered cash flow anaiysis and projrct ranking were the most difficult processes. Liberator found out that the respondents had encountered difficulties on the diversity of project types, and the uncertainty of data.

Therefore, the purpose of this study is trytng to build the prototype of an expert system which can support the decision makers in solving diverse capital budgeting problems without suffering the difficulties in the solving process. In this study, we focus on two major constructions in the proposed expert system: the knowledge base and the inference system. We understand that this study is not intensive enough for a real expert system. However, we would like to initiate the idea and pave the way for a prospective future expert system.

1.4 Literature Survey

One of the knowledger engineer's tasks is to understand thoroughly and systematically the details of this topic. We have seen from the literature quite a iot of theories, methods and applications regarding the capitai budgeting topic. The key words we used in surveying the literature are : finance, capital budgeting, portfolio theory, operations

research, mathematical programmming and exper system.

Table 1.3 lists the journals which were surveyed. We went through all these journals published in the past 10 years (1977 - 1988). Table 1.4 lists the literature survey on textbooks regarding this topic. Table 1.5 lists references associated with the major contents in the thesis.

Table 1.3 Literatures survey on journals

1. Financial Analysis Journal
2. Financial Management
3. Journal of Finance
4. Journal of Finance & Quantitative Analysis
5. Financial World
6. Computer & Operations Research
7. Operations Research
8. AIIE Transactions
9. Management Sciences
10. Interface
11. Decision Science

Table 1.4 Literatures survey on textbooks

Key words	Reference
Finance and Capital Budgeting	Bussey 1978, Clark 1979, Crum 1981, Levy 1977, Osteryoung 1979, Townsend 1969, White 1984
Portfolio Theory	Elton 1981, Harrington 1981, Maginn 1983, Mittra 1981
Operations Research , Math. Programming	Hwang & Masud 1979, Hwang & Yoon 1981, Ignizio 1976, Lee 1972, McMillan i975, Zeleny 1974
Expert System	Feigenbaum 1983 Hwang 1987, Waterman 1986

Table 1.5 Reference summary

Subject	Reference
Surveys on capital budgeting practices	[23], [26]
Project selection/ ranking	[4], [5], [6], [7], [8], [19], [31], [36], [37], [39], [42], [43]
Portfolio theory	[9], [13], [27], [28], [30], [34], [36]
Math. programming	[1], [2], [11], [12], [15], [16], [18], [20], [21], [23], [29], [33], [35], [38], [44]
Expert system	[3], [10], [14], [17], [24], [32], [41]

1.5 Thesis Structure

There are five chapters in the thesis. The first chapter, the introduction, introduces the concepts of capital budgeting and the expert system. It also declares the purpose of the study and reviews the literature survey.

The second chapter discusses the fundamental concepts of cash flows and compounding. The concepts of hurdle rate, cost of capital, and minimum attractive rate of return are discussed. It also covers compounding and series of cash flows.

The third chapter is the knowledge base construction which covers capital budgeting techniques in detail. Capital

budgeting techniques are classified into three main categories: project selection and ranking (under certainty, risk or uncertainty), portfolio selection and mathematical programming. Most of the methods are discussed in detail with examples.

The fourth chapter is the inference system construction. In this chapter we analyze the facts of problems and the assumptions and conditions underlying each solution method. We have defined the rules of inference procedures which may connect the stated problem with specific solutions. There are four major modules in the inference system: project evaluation, project ranking, portfolio selection and mathematical programming.

Chapter 5 is the summary and conclusion of the thesis.

Chap 2. FUNDAMENTAL CONCEPTS --- CASH FLOW AND COMPOUNDING

2.0 Introduction

The concepts and mathematics of cash flow and compounding are widely discussed in capital budgeting or engineering economics textbooks. The purpose of this chapter is to try to review briefly these techniques since they are fundamental to the application of capital budgeting decision making. The contents of this chapter follow closely those from Bussey [4], Clark [5], Osteryoung [31] and White [42].

2.1 Cash Flows and Discounted Cash Flows

The analysis of capital investment is primarily based on the knowledge of cash outflows to acquire assets and the cash inflows that are expected from the investments. The cash flow profile is therefore a characteristic used to compare the economic performance among projects. Please note that the cash flow (CF) we use in the study always refers to after-tax cash flow (ATCF). ATCF is the amount remaining after income taxes and deductions, including interest but excluding depreciation allowance.

An important concept underlies most of the multiple period cash flows is that cash has time value. Many factors affect the time value, such as inflation, interest rate, return on

investment and so on. The following example is intended to illustrate the time value of money.

Example 2.1

A depositor puts \$3,000 in a bank at 5% interest, compounded annually for 3 years. The amount which will be accrued at the end of 3 years is :

$$S_3 = \$3,000 (1 + .05)^3$$

$$= \$3,472.88$$

This is a very simple example of time value due to the interest and the compounding.

Example 2.2

Two alternative projects both involve an investment of \$1,000 in value that lasts for 4 years. Their profiles are given in the following table:

Table 2.1 Cash flow profiles for two investment alternatives

End of year (EOY)	Project A	Project B	difference (A - B)
0	-\$10,000	-\$10,000	\$0
1	+ 7,000	+ 1,000	+6,000
2	+ 5,000	+ 3,000	+2,000
3	+ 3,000	+ 5,000	-2,000
4	+ 1,000	+ 7,000	-6,000

Project A invests in minicomputer software. Because the

competition is expected to be very high, a decline revenue profile is anticipated. Project B investS in a land development venture, different parcels of land are to be sold over 4 years. Since the land is anticipated to increase in value, an increasing revenue profile is expected. Each project results in a net cash flow of $6,000 over its useful life. Which project would you prefer ? The $6,000 and the $2,000 difference at the end of the first two years are worth more than the $6,000 and the $2,000 difference at the end of the last two years. Project A is better than project B. If youchoose project B, then you ignore the time value of cash flows.

2.2 Discount Rate and Compounding

The discounted-cash-flow approach used to evaluate alternative projects measures cash flows in terms of a required discount rate (hurdle rate) to determine their acceptability. The hurdle rate is referred to the firm's cost of capital. It represents the cost of funds used to acquire the total assets of the firm. In White's [42] text, *Minimum Attractive Rate of Return* (MARR) is used as the hurdle rate, while in Clark's [05], Bussey's [04] and Osteryoung's [31] texts , *cost of capital* is used as the hurdle rate.

2.2.1 Cost of Capital

The cost of capital refers to the rate of return expected by those parties contributing to the financial structure: creditors and preferred and common shareholders. It represents the cost of funds used to acquire the assets of the firm. Thus, it is generally calculated as a weighted-average marginal cost associated with each type of capital included in the financial structure of the enterprise.

Several factors merit additional comments [5]:

1. The cost of capital, considered as a rate of return, disaggregates into risk-free rate plus premium of risk.
2. The cost of capital represents a rate of return that will maintain the market value of the outstanding securities within the context of overall market movements.
3. The cost of capital is the rate which will enable the firm to sell new securities at current market price level.

Even though there are different ways to interpret cost of capital, there is an underlying commonality: <u>The firm select</u> <u>capital projects with the goal of obtaining a yield at least sufficient to cover its cost of capital</u>. Normally, the cost of capital to a firm is decided by the weighted marginal cost of its financial structure mix, which may comprise

equity, stock, debt, retained earnings, and so forth. The details are not meant to be covered here. Interested readers may refer to Bussey's [4] or Clark's [5] text for more understanding.

2.2.2 Compounding

Normally, the discount rate is defined on the basis of end-of-year compunding and on end-of-year cash flows. What will be the equivalent rate if the period is quarterly or monthly or even continuous compounding instead of annual compounding ?

2.2.2.1 Multiple Compounding in a Year

If $1,000 is borrowed at 12% interest, compounded quarterly, then the amount owed at the end of the first year (4 periods) is :

$$F = \$1,000\ (1 + .03)^4$$
$$= \$1,125.51$$

We may see that the equivalent annual rate is:

$$i = (1125.51 - 1000) \,/\, 1000$$
$$= 12.55\%$$

which is higher than 12%.

In this case, the rate 12% is referred to as <u>the nominal annual rate</u>; the rate 12.55% is referred to as <u>the effective annual rate</u>. The relationship between the effective rate and

the nominal rate is as follows:

$$i = \left(1 + \frac{r}{n} \right)^{n} - 1 \qquad (2.1)$$

where i : the effective rate of a year

r : the nominal rate of a year

n : number of periods in a year

Example 2.3

A $1,000 deposit has the rate of 12% compounded monthly. What will be the amount accrued by the depositor after 2 years?

Solution :

$$i = \left(1 + \frac{.12}{12} \right)^{12} - 1$$

$$= 12.68\%$$

$$F = \$1,000\ (1 + 0.1268)^{2}$$

$$= \$1,269.68$$

2.2.2.2 **Continuous Compounding**

Monetary transactions occur daily or hourly in most business, and money is normally put to work as soon as it is received. Compounding , in this sense, is occurring quite frequently. For such rapid compounding, continuous compounding relations should be used. **This means that there are an infinite number of periods in a year. Mathematically,**

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \qquad (2.2)$$

when compounded infinitely, the relationship between effective rate and nominal rate can be expressed as follows:

$$i = \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n - 1$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n/r}\right)^{(n/r)r} - 1$$

Since r is independent of the equation, therefore

$$i = \left[\lim_{n \to \infty} \left(1 + \frac{1}{n/r}\right)^{n/r}\right]^r - 1$$

Letting $h = n/r$, we may see $h \to \infty$ as $n \to \infty$, so,

$$i = \left[\lim_{h \to \infty} \left(1 + \frac{1}{h}\right)^h\right]^r - 1$$

$$= e^r - 1 \qquad (2.3)$$

<u>Example 2.4</u>

If \$2,000 is invested in a fund that pays interest at a rate of 12% compounded continuously. What will be the cumulative amount after 5 years?

Solution: $i = e^r - 1$

$= 12.75\%$

therefore, $F_5 = \$2{,}000\ (1 + 0.1275)^5$

$= \$3{,}644.29$

2.3 Single Sum of Money

--- Future Value (FV) and Present Value (PV)

In constructing cash flow profiles, it will be easier for project analyst to make decisionsif the whole cash flow can be summarized into a single sum. Two kinds of single sums of money are frequently used in project cash flow analysis, the present value (<u>PV</u>) and the future value (<u>FV</u>).

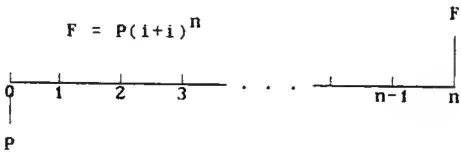


Figure 2.1 Cash flow diagram of the relationship between PV and FV

In considering the time value of money, it is convenient to represent mathematically the relationship between the current present value of a single sum of money and its future value. Their relationship can be seen from Figure 2.1.

Suppose we deposit P amount in the bank with interest rate i, then the future value F after n years can be expressed as follows:

$F = P\ (1 + i)$ (2.4)

The quantity $(1 + i)^n$ is referred to as <u>the single sum, future worth factor</u> and is denoted <u>(F¦P i%, n)</u>.

So, $F = P\ (1 + i)^n$

$= P\ (F|P\ i\%,\ n)$ (2.5)

<u>Example 2.5</u>

An individual borrows \$1,000 at 12% compounded **annually.** The debt will be paid back after 5 years. How much should be repaid?

Solution:

$F = P\ (F|P\ 12\%,\ 5)$

$= \$1{,}000\ (1 + .12)^5$

$= \$1{,}762.30$

It is also a very simple matter to determine a single sum of present value (PV) when given future values, discount rate i and period n. Since

$F = P\ (1 + i)^n$ (2.4, recalled)

therefore,

$P = F\ (1 + i)^{-n}$ (2.6)

or

$P = F\ (P|F\ i\%,\ n)$ (2.7)

where $(1 + i)^{-n}$ and (P|F i%, n) are referred to as the single sum present worth factor.

Example 2.6

If you wish to accumulate \$10,000 in a saving account 4 years from now, and the account pays interest at a rate of 9%, compounded annually. How much must be deposited today?

Solution:

$$P = F\ (P|F\ 9\%,4)$$
$$= \$10{,}000\ (1 + .09)^{-4}$$
$$= \$7{,}084.25$$

2.4 Series of Cash Flows

Having considered the transformation of a single sum of money, we generalize our discussion to consider the conversion of a series of cash flows to present worth and future worth equivalents.

Let A_t denote the magnitude of a cash flow at the end of period t .

The present worth equivalent for the cash flow series is equal to the sum of the present worth equivalents for the individual cash flow. The following figure (Figure 2.2) may express their relationship.

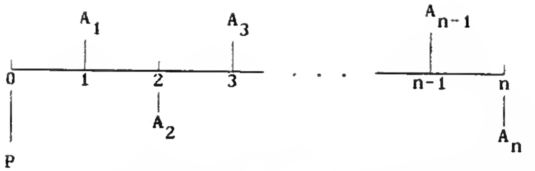


Figure 2.2 Cash flow diagram of relationship between present worth and series of cash flows

Consequently,

$$P = A_1(1 + i)^{-1} + A_2(1 + i)^{-2} + \ldots + A_n(1 + i)^{-n}$$

or using the summation notation,

$$P = \sum_{t=0}^{n} A_t(1+i)^{-t} \qquad (2.8)$$

or

$$P = \sum_{t=0}^{n} A_t(P|F\ i\%,\ t) \qquad (2.9)$$

Similarly, the future worth equivalent is equal to the sum of the future equivalents for the individual cash flow. The following figure (Figure 2.3) may express their relationship.

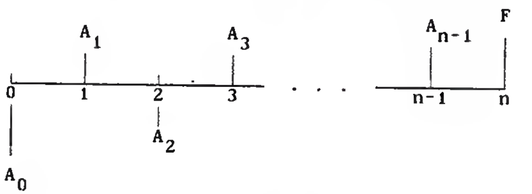


Figure 2.3 Cash flow diagram of relationship between future worth and series of csah flows

Thus,

$$F = A_1(1 + i)^{n-1} + A_2(1 + i)^{n-2} + \ldots + A_{n-1}(1 + i) + A_n$$

or using the summation notation,

$$P = \sum_{t=0}^{n} A_t(1+i)^{n-t} \qquad (2.10)$$

or

$$P = \sum_{t=0}^{n} A_t(F|P\ i\%,\ t) \qquad (2.11)$$

<u>Example</u> <u>2.7</u>

A series of cash flows are depicted in Figure 2.4 as follows:

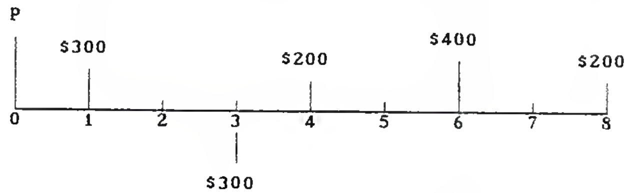


Figure 2.4 Series of cash flows

Using an interest rate of 6% per period, the present worth equivalent at period 0 is given by

$$\begin{aligned} P &= \$300(P|F\ 6\%,\ 1) - \$300(P|F\ 6\%,3) + \$200(P|F\ 6\%,4) \\ &\quad + \$400(P|F\ 6\%,6) + \$200(P|F\ 6\%,8) \\ &= \$300(1.06)^{-1} - \$300(1.06)^{-3} + \$200(1.06)^{-4} \\ &\quad + \$400(1.06)^{-6} + \$200(1.06)^{-8} \\ &= \$597.02 \end{aligned}$$

Similarly, the equivalent future worth at the end of period 8 is given by

$$
\begin{aligned}
F &= \$300(F|P\ 6\%,\ 7) - \$300(F|P\ 6\%,5) + \$200(F|P\ 6\%,4) \\
&\quad + \$400(F|P\ 6\%,2) + \$200 \\
&= \$300(1.06)^7 - \$300(1.06)^5 + \$200(1.06)^4 \\
&\quad + \$400(1.06)^2 + \$200 \\
&= \$951.56
\end{aligned}
$$

2.4.1 Uniform Series of Cash Flows

A uniform series of cash flows exists when all of the cash flows in the series are equal. In the case of a uniform series, the present worth equivalent is given by

$$P = \sum_{t=1}^{n} A(1 + i)^{-t} \qquad (2.12)$$

where A is the uniform magnitude of an individual cash flow in the series.

Letting $X=(1+i)^{-1}$ and bringing A out of the summation yields

$$P = A \sum_{t=1}^{n} X^t \qquad (2.13)$$

Eq. (2.13) represents A in terms of a geometric series. The summation of such series is given by

$$X + X^2 + \ldots + X^n = \frac{X(1 - X^n)}{1 - X} \qquad (2.14)$$

therefore,

$$P = \frac{AX(1 - X^n)}{1 - X} \qquad (2.15)$$

replacing X with $(1 + i)^{-1}$ yields

$$P = A\left[\frac{(1 + i)^n - 1}{i(1 + i)^n}\right] \qquad (2.16)$$

Let us define the term

$$(P|A\ i\%,\ n) = \frac{(1 + i)^n - 1}{i(1 + i)^n} \qquad (2.17)$$

then

$$P = A(P|A\ i\%,\ n) \qquad (2.18)$$

where <u>(P|A i%, n)</u> is referred to as <u>uniform series, present worth factor</u>. The reciprocal relationship betwwn P and A can be expressed as

$$A = P\left[\frac{i(1 + i)^n}{(1 + i)^n - 1}\right] \qquad (2.19)$$

Let us define the term

$$(A|P\ i\%,\ n) = \frac{i(1 + i)^n}{(1 + i)^n - 1} \qquad (2.20)$$

then

$$A = P(A|P\ i\%,\ n) \qquad (2.21)$$

where <u>(A|P i%, n)</u> is called <u>capital recovery factor</u>.

<u>Example 2.8</u>

An individual wishes to deposit a single sum of money in a

saving account so that five equal withdrawals of $2,000 can be made before depleting the fund. The first withdrawal is expected to occur 1 year after the deposit and the interest rate is 12%, compounded annually.

(1) How much should be deposited?

(2) Suppose the first withdrawal will not occur until 3 years after the **deposit. How** much should he deposit ?

Solution:

(1) The cash flow diagram is as follows

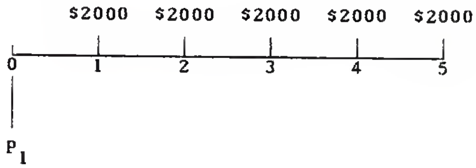


Figure 2.5 Cash flow diagram

From Figure 2.5, we may get

$$P_1 = \$2,000(P|A\ 12\%,\ 5)$$
$$= \$7,209.60$$

(2) With a lead time of 3 years for the first withdrawal, cash flow diagram should be revised as follows:

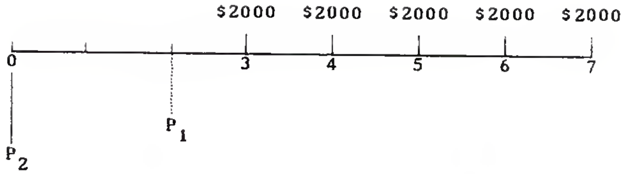


Figure 2.6 Cash flow diagram

The value P_1 we obtained in case (1) becomes the cash inflow of period 2 in this case. Therefore

$$P_2 = A(P|A\ 12\%,\ 5)(P|F\ 12\%,\ 2)$$

$$= \$7{,}209.6(0.7972)$$

$$= \$\ 5{,}747.49$$

Similarly, the future worth of a uniform series is obtained by

$$F = P\ (1 + i)^n \qquad (\ 2.4,\ \text{recalled}\ )$$

where

$$P = A\left[\frac{(1 + i)^n - 1}{i(1 + i)^n}\right] \qquad (\ 2.16,\ \text{recalled})$$

therefore,

$$F = A\left[\frac{(1 + i)^n - 1}{i}\right] \qquad (\ 2.22\ )$$

Defining

$$(F|A\ i,\ n) = \frac{(1 + i)^n - 1}{i} \qquad (\ 2.23\ )$$

we may get

$$F = A(F|A\ i\%,\ n)$$

where (F|A i%, n) is referred to as the uniform series, future worth factor. The reciprocal relationship between A and F can be easily obtained from Eq. (2.22).

We find out that

$$A = F\left[\frac{i}{(1+i)^n - 1}\right] \qquad (2.24)$$

Defining

$$(A|F\ i\%,\ n) = \frac{i}{(1+i)^n - 1} \qquad (2.25)$$

we get

$$A = F(A|F\ i\%,\ n) \qquad (2.26)$$

where (A|F i%, n) is referred to as the sinking fund factor .

Example 2.9

If annual deposits of $1,000 are made into a saving account for 30 years, how much will be in the fund immediately after the last deposit if the interest rate is 8% compounded annually?

Solution :

$$F = \$1000(F|A\ 8\%,\ 30)$$

$$= \$1,000(113.2831)$$

$$= \$113,283.10$$

2.4.2 Gradient Series of Cash Flows

A gradient series cash flows occurs when the value of a given

cash flow is greater than the value of the previous cash flow by a constant amount, say G. The series can be represented by the sum of a uniform series and a gradient series depicted in Figure 2.7. The size of the cash flow in the gradient series occurring at the end of period t is given by,

$$A_t = (t - 1)G \qquad t = 1, 2, \ldots, n \qquad (2.27)$$

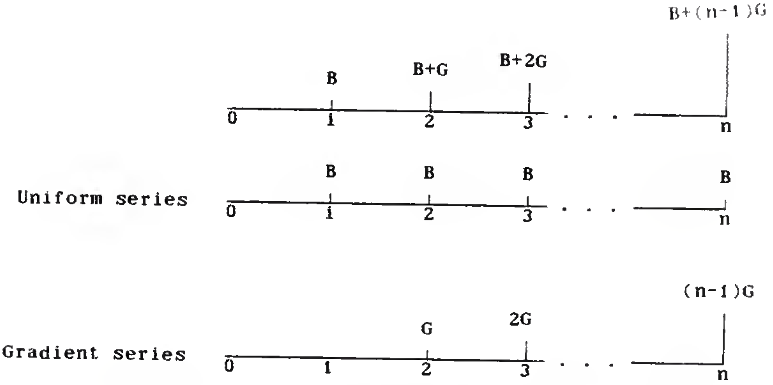


Figure 2.7 Cash flow diagram of a combination of uniform and gradient series

The present worth equivalent of a gradient series is obtained by

$$P = \sum_{t=0}^{n} A_t(1 + i)^{-t} \qquad (2.8 \text{ recalled})$$

$$= G(1+i)^{-2} + 2G(1+i)^{-3} + \ldots + (n-1)G(1+i)^{-n} \qquad (2.28)$$

Now, multiply both sides by $(1 + i)^n$, yields

$$P(1+i)^n = G(1+i)^{n-2} + 2G(1+i)^{n-3} + \ldots + (n-2)G(1+i) + (n-1)G$$

Note that this sequence can be recomposed by factoring into the following sequence

$$\begin{aligned}
P(1+i)^n = {} & G[(1+i)^{n-2} + (1+i)^{n-3} + (1+i)^{n-4} + \ldots + (1+i)^2 + (1+i) + 1\,] \\
& + G[(1+i)^{n-3} + (1+i)^{n-4} + \ldots + (1+i)^2 + (1+i) + 1\,] \\
& + G[(1+i)^{n-4} + \ldots + (1+i)^2 + (1+i) + 1\,] \\
& + \ldots + G[(1+i)^2 + (1+i) + 1\,] \\
& + G[(1+i) + 1\,]
\end{aligned}$$

$$= G(F|A\ i,\ n-1) + G(F|A\ i,\ n-2) + \ldots + G(F|A\ i,\ 1)$$

$$= G\left[\frac{(1+i)^{n-1} - 1}{i} + \frac{(1+i)^{n-2} - 1}{i} + \ldots + \frac{(1+i)^2 - 1}{i} + \frac{(1+i) - 1}{i}\right]$$

$$= \frac{G}{i}\left[(1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i)^2 + (1+i) - (n-1)\right]$$

If we move the nG/i outside the brackets, then

$$P(1+i)^n = \frac{G}{i}\,[(1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i) + 1] - \frac{nG}{i} \qquad (2.29)$$

The terms inside the bracket in Eq. (2.29) define the (F|A i%, n) factors for n periods; hence Eq. (2.29) becomes

$$P(1+i)^n = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i}\right] - \frac{nG}{i} \qquad (2.30)$$

$P(1+i)^n$ is simply the n_{th} period future value. In order to find

an equivalent end-of-period amount in an equal series, A, we multiply Eq. (2.30) on both sides by the (A|F i%, n) factor:

$$A = F(A|F\ i\%,\ n) = P(1+i)^n (A|F\ i\%,\ n) = P(1+i)^n \left[\frac{i}{(1+i)^n - 1} \right]$$

$$= \left\{ \frac{G}{i} \left[\frac{(1+i)^n - 1}{i} \right] - \frac{nG}{i} \right\} \left[\frac{i}{(1+i)^n - 1} \right]$$

$$= \frac{G}{i} - \frac{nG}{(1+i)^n - 1}$$

$$= G\left[\frac{i}{i} - \frac{n}{(1+i)^n - 1} \right] \qquad (2.31)$$

Defining

$$(A|G\ i\%,\ n) = \frac{1}{i} - \frac{n}{(1+i)^n - 1} \qquad (2.32)$$

we may get

$$A = G(G|A\ i\%,\ n)$$

where the factor <u>(G|A i%, n)</u> is called <u>the gradient-to-uniform series conversion factor</u>. It converts a series of uniformly increasing (or decreasing) end-of-period amounts into an equivalent series of equal end-of-period amounts. Note that the gradient conversion factor is derived from an assumption of zero cash flow at the beginning of the gradient.

Recalling Eq. (2.30), if we divide $(1+i)^n$ on both sides, the equation becomes

$$P = \frac{G}{i} \left[\frac{(1+i)^n - 1}{i(1+i)^n} \right] - \frac{nG}{i(1+i)^n} \qquad (2.33)$$

or,

$$P = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i} - n\right]\left[\frac{1}{(1+i)^n}\right] \qquad (2.34)$$

Let us denote

$$(P|G\ i\%,\ n) = \frac{1}{i}\left[\frac{(1+i)^n - 1}{i} - n\right]\left[\frac{1}{(1+i)^n}\right] \qquad (2.35)$$

then

$$P = G(P|G\ i\%,\ n)$$

where the factor <u>$(P|G\ i,\ n)$</u> is referred to as <u>the gradient series, present worth factor.</u>

<u>Example 2.10</u>

Supposing that certain end-of-year expenses are estimated as follows:

End of year	Expense
1	\$3,000
2	5,000
3	7,000
4	9,000

If the effective interest rate is 15%, what is the equivalent annual end-of-year amount?

Solution:

The initial amount at $t = 1$ is B = \$3,000 and the constant difference is G = \$2,000 per period. Therefore

$A = B + G(A|G\ 15\%,\ 4)$

$= \$3,000 + \$2,000(1.3263)$

$= \$5,652.6$

2.4.3 Geometric Series

The geometric cash flow series, as depicted in Figure 2.8, occurs when the size of cash flow increases (decreases) by a fixed percent from the previous cash flow.

Let j denote the percent change in size, then the size of t^{th} period cash flow can be given by

$$A_t = A_{t-1}(1 + j) \quad t = 2, 3, \ldots, n \qquad (2.36)$$

or

$$A_t = A_1(1 + j)^{t-1} \quad t = 2, 3, \ldots, n \qquad (2.37)$$

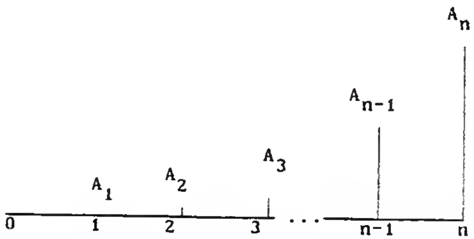


Figure 2.8 Cash flow diagram of the geometric series

The present worth equivalent of the geometric series can be obtained by substituting Eq. (2.37) into Eq. (2.8) to obtain

$$P = \sum_{t=1}^{n} A_1(1+j)^{t-1}(1+i)^{-t} \qquad (2.38)$$

or

$$P = A_1(1+j)\sum_{t=1}^{n}\left[\frac{1+j}{1+i}\right]^t \qquad (2.39)$$

It is easy to derive the following results:

$$P = \frac{nA_1}{1+i} \qquad \text{when } i = j$$

$$= A_1\left[\frac{1-(1+j)^n(1+i)^{-n}}{i-j}\right] \qquad \text{when } i \neq j$$

or

$$P = A_1(P|A\ i,\ j,\ n) \qquad (2.41)$$

where the factor <u>(P|A i%, j%, n)</u>is <u>the geometric series, present worth factor.</u>

<u>Example 2.11</u>

Labor cost have been increasing at an annual rate of 8%. A firm wishes to set aside funds to cover labor costs for the next five years. How much must be set aside today if the money will be invested and will earn interest at the rate of 10%, given that labor costs will be $50,000 next year ? ·

olution:

For the problem $A_1 = 50000$, $i = 10\%$ and $j = 8\%$, the present worth equivalent is

$$P = A\ (P|A_1\ 10\%,\ 8\%,\ 5)$$

$$= \$50{,}000(4.3831)$$

$$= \$219{,}155$$

Chap 3. Knowledge Base Construction

--- Capital Budgeting Techniques

3.0 Introduction

In order to offer project analysts better understanding of the quantitative capital budgeting techniques, **most of these** techniques were reviewed and included in this chapter. These techniques also **constitute the** basic knowledge base of a capital budgeting expert system. Literature surveys through 1970's and 1980's textbooks [4], [5], [6], [9], [13], [15], [16], [17], [18], [23], [25], [27], [29], [30], [31], [39], [40], [42], [44] and journals [2], [11], [12], [19], [20], [26], [34], [36], [37], [39] **have** contributed to the **development of this chapter.**

In this chapter, the capital budgeting techniques are classified into three categories: project evaluation and ranking (under certainty, risk or uncertainty), portfolio selection and mathematical programming.

3.1 Project Evaluation and Ranking --- Under Certainty

Certainty means the outcome of each investment **is** completely known in advance. Although quite unrealistic, there are considerable advantages in making such **assumptions : the** simplicity of problem and isolation of errors. Traditional

techniques may be classified into two categories: cash flow approach and discounted cash flow approach.

3.1.1 Cash Flow Approach

For this approach, time value of money is ignored. It means the project analyst assumes that the cost of capital is almost negligible. Two methods are covered: the Payback (PB) method and the Return on Investment (ROI) method.

3.1.1.1 Payback Period (PB) Method

The payback period is defined as the number of years required to recover the investment of a project. It is actually a measure of a project's liquidity and capital recovery rate. When using PB, projects may be accepted or rejected based on the number of years to recover their cost. Mathematically, let m denote the smallest value which satisfies the following equation:

$$\sum_{t=1}^{m} R_t \geq C \qquad (3.1)$$

Where C : the net cost

R_t: the net revenue at period t

<u>Example 3.1</u>

A corporation plans to invest funds to purchase a new machine.

The projected cash flows are shown below. Determine the PB for this investment.

Period	Expected cash flow
0	$-10,000
1	-4,000
2-6	3,000
7-15	6,000

Solution: The PB may be found using the following table:

Period	Cash Flow	Net Cash Flow
0	$-10,000	$-10,000
1	-4,000	-14,000
2	3,000	-11,000
3	3,000	-8,000
4	3,000	-5,000
5	3,000	-2,000
6	3,000	1,000

$$PB = 5 + (2000/3000) = 5\tfrac{2}{3} \text{ years}$$

The PB method is used widely in the industry. Various reasons are given for this [4], [5]:

1. It is simple.
2. It does not require a discount rate.
3. It provides a rough measure of the liquidity of an investment.
4. Managers with conservative attitude like this measure.
5. Projects with risks growing with the time horizon are be appropriate for evaluation by the PB measure.

The method also has been discredited because of the following shortcomings:

1. It fails to consider the cash flows beyond the PB.

2. It fails to consider the time value.

3. It does not differentiate between projects requiring different investments.

Clark recommended that the PB should be used as a supplemental tool in conjunction with discounted cash flow analysis.

3.1.1.2 Return On Investment (ROI) Method

The return on investment measure compares the yearly after-tax (or pre-tax) income with the investment in the assets. There are four methods commonly used.

Method 1: Average Return on Investment

$$= \frac{\text{annual income}}{\text{original investment}} \times 100\%$$

Method 2: Annual Return on Average Investment

$$= \frac{\text{annual income}}{\text{original investment} \div 2} \times 100\%$$

Method 3: Average Return on Average Investment

$$= \frac{\text{Total income} - \text{original investment}}{\frac{\text{original investment}}{2} \times \text{years}} \times 100\%$$

Method 4: Average Book Return on Investment

$$= \frac{\text{Total income} - \text{original investment}}{\text{weighted average investment}} \times 100\%$$

The weighted average investment is the sum of book values of each year according to straight-line depreciation over project life.

<u>Example 3.2</u>

An investment of $1,000 on a bench lathe is expected to have a cash flow profile as follows:

Investment		\$1,000
estimate useful life		5 years
Income	year 1	\$300
	2	300
	3	300
	4	300
	5	300
		1,500

The return on investment computations are:

1) Annual Return on Investment $= \dfrac{300}{1000} \times 100\% = 30\%$

2) Annual Return on Avg. Inv. $= \dfrac{300}{1000 \div 2} \times 100\% = 60\%$

3) Avg. Return on Avg. Inv. $= \dfrac{1500 - 1000}{\frac{1000}{2} \times 5} \times 100\% = 20\%$

4) Avg. Book Return on Inv.

$$= \frac{1500 - 1000}{\frac{1000+800+600+400+200}{5} \times 5} \times 100\% = 16\tfrac{2}{3}\ \%$$

Please note that the project analyst should be consistent with the method chosen to evaluate alternatives in order to have consistent results.

The shortcomings of the rate of return procedures are [5]:

1. The time value is not considered.
2. The book values of capital are used instead of the market value of capital. Since the real market situation is not reflected, the ROI measure may be extremely misleading.

Though the method is widely used, we do not recommend it . The ROI measures offer only rough information about project's profitability. It should be used as a supplemental tool in addition to other techniques.

3.1.2 Discounted Cash Flow Approach

As we have discussed in Chap. 3, the discounted cash flow approach takes the time value of money into consideration. In this section we will illustrate six main measures of investment worth that were chosen either because they are currently used in practice or because they have good theoretical arguments. These measures are (1) Net Present Value (NPV), (2) Net Future Value (NFV), (3) Annual Worth (AW), (4) Internal Rate of Return (IRR), (5) External **Rate** of Return (ERR) and (6) Profitability Index (PI).

3.1.2.1 Total Cash Flow Approach and Incremental Cash Flow Approach

There are two basic concepts used in comparing mutually exclusive alternatives when the six different measures are employed. One involves use of the **total cash flow approach** in which the total cash flows associated with each alternative are considered individually. The other involves use of the **incremental cash flow approach** in which alternatives are

compared pairwise, and only the incremental cash flow are considered. Both methods are correct and will yield consistent results if performed correctly.

3.1.2.1.1 Total Cash Flow Approach

With this approach, we calculate for each alternative the measure of investment worth not only from the investment on **projrct, but also from the entire budget available.** ***We assume*** *that the remaining budget is invested at the cost of capital.* The following example is intended to explain this approach. Please note that the 'Do Nothing' alternative is always considered an alternative.

<u>Example 3.3</u>

Two mutually exclusive alternatives have cash flow profile given as follows: (assuming 15% cost of capital)

Table 3.1 Cash Flow Profile of alternatives

	Net Cash Flows		
EOY	Alternative 0	1	2
0	0	\$-50,000	\$-75,000
1	0	20,000	20,000
2	0	20,000	25,000
3	0	20,000	30,000
4	0	20,000	35,000
5	0	20,000	40,000

It is evident from table 3.1 that at least \$75,000 is available. The total cash flow approach assumes that the

remaining budget can be reinvested at the rate of 15% over 6 years. For alternative 1, the **remaining** budget is \$25,000 (\$75,000 − 50,000). The annual worth of the investment is

$$A = \$25,000\ (A|P\ 15\%,\ 5) = 25000(0.298316)$$
$$= \$7,457.9$$

The following table shows the cash flow profile of alternative 1 (A_1) with total cash flow approach.

Table 3.2 Total Cash Flow Approach (A_1)

EOY	CF of A_1	Investment	TCF
0	\$-50,000	\$-25,000.0	\$-75,000.0
1	20,000	7,457.9	27,457.9
2	20,000	7,457.9	27,457.9
3	20,000	7,457.9	27,457.9
4	20,000	7,457.9	27,457.9
5	20,000	7,457.9	27,457.9

The Net Present Value of A_1 is :

$$PW_1 = -75,000 + 7,457.5(P|A\ 15\%,\ 5)$$
$$= \$17,043.1$$

When we ignore the investment of the remaining budget, the Net Present Value of A_1 is:

$$PW_1' = -50,000 + 20,000(P|A\ 15\%,\ 5)$$
$$= \$17,043.1$$

which is identical with PW_1.

The reason is that the net present value of \$25,000 investment equals to zero with the rate of 15%. Therefore, when using NPV, FV or AW measures, we need only to measure the cash flows

due to each alternative. But this is not true for ERR, IRR or PI measure, because they have different assumptions on the use **of the remaining budget.**

The NPVs for alternative 0 and 2 are:

$$PW_0 = 0$$

$$\begin{aligned} PW_2 &= -75{,}000 + 20{,}000(P|A\ 15\%,\ 5) + 5{,}000(P|G\ 15\%,\ 5) \\ &= -75{,}000 + 20{,}000(3.35216) + 5{,}000(5.77514) \\ &= \$20{,}918.9 \end{aligned}$$

The ranking order by Total Cash Flow approach is A_2 , A_1.

3.1.2.1.2 Incremental Cash Flow Approach

With incremental cash flow approach, we calculate the measure of investment worth for the incremental cash flows between pairs of alternatives. The sets of mutually exclusive alternatives are usually ordered from lowest to highest initial investment. Let us illustrate the approach by the following example.

Example 3.4

From Table 3.1, the incremental cash flow Profile is as.

Table 3.3 Cash Flow Profile of alternatives

		Net	Cash	Flows	
EOY	Alt. 0	1	2	1 - 0	2 - 1
0	0	\$-50,000	\$-75,000	\$-50,000	\$-25,000
1	0	20,000	20,000	20,000	0
2	0	20,000	25,000	20,000	5,000
3	0	20,000	30,000	20,000	10,000
4	0	20,000	35,000	20,000	15,000
5	0	20,000	40,000	20,000	20,000

Between alternative 1 and 0

$$PW_{1-0} = PW_1 = \$17{,}043.1 > 0$$

We prefer alternative 1 to 0.

Between alternative 2 and 1

$$\begin{aligned} PW_{2-1} &= -25{,}000 + 5{,}000(P|G\ 15\%,\ 5) \\ &= -25{,}000 + 5{,}000(5.7751) \\ &= 3{,}875.5 > 0 \end{aligned}$$

We prefer alternative 2 to 1.

Therefore, the ranking order should be A_2, A_1, A_0.

Figure 3.1 and 3.2 (by White [42]) show the decision rules for the six measures using different approaches. White recommends use of the incremental cash flow approach since it is not only correct but also much easier.

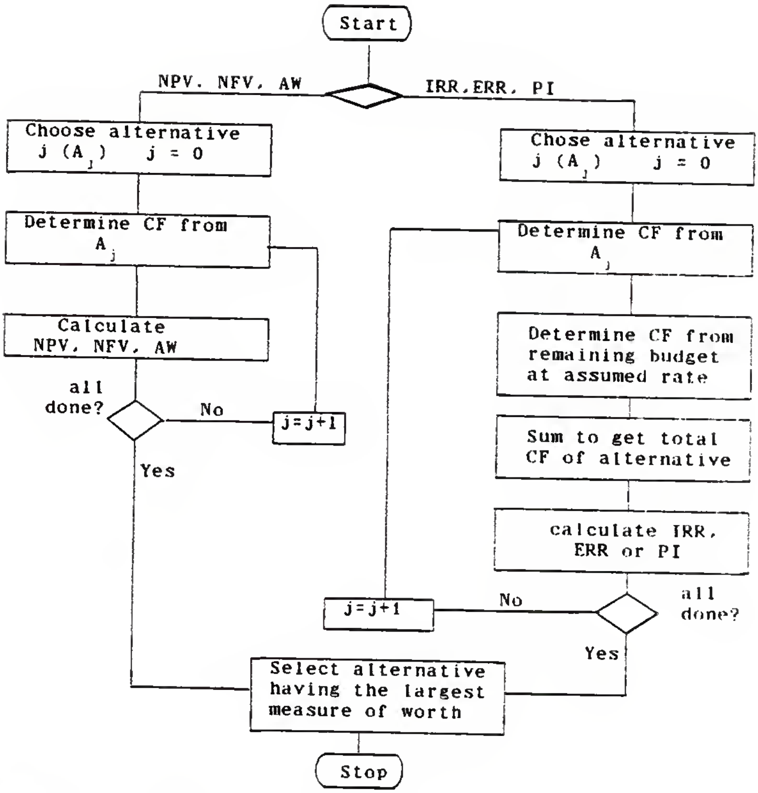


Figure 3.1 Flowchart of Total Cash Flow Approach [42]

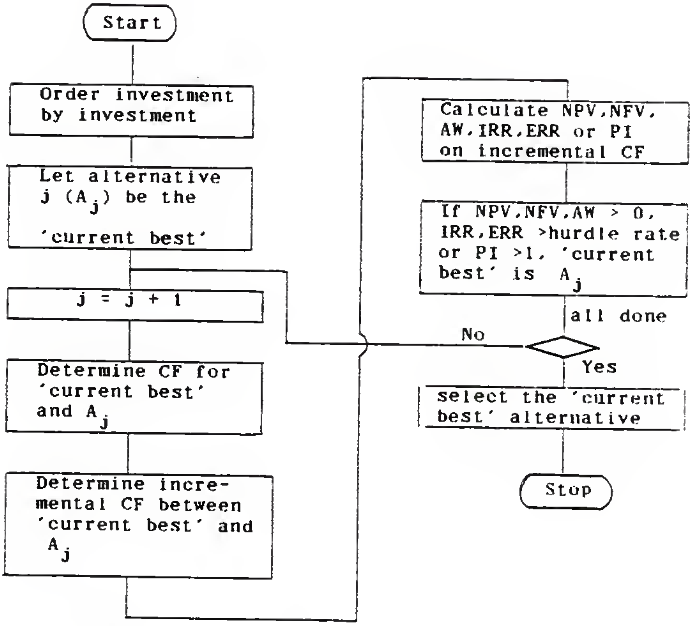


Figure 3.2 Flowchart of Incremental Cash Flow Approach [42]

3.1.2.2 Net Present Value (NPV), Net Future Value (NFV) and Annual Worth (AW) Methods

The NPV, NFV and AW are measures indicating the performance of projects. While ranking projects, these measures always yield consistent results. In practice, NPV measure is used more

frequently than NFV and AW,

The form for expressing NPV was developed in Chap. 2. Let us repeat the expression here:

$$P = \sum_{t=0}^{n} A_t(1 + i)^{-t} \qquad (\text{ 2.8, recalled })$$

More generaly, when i varies by period, denoted as i_t, the general form can be expressed by:

$$P = \sum_{t=0}^{n} A_t(1 + i_t)^{-t} \qquad (\text{ 3.2 })$$

Similarly, the general form of NFV can be expressed by:

$$F = \sum_{t=0}^{n} A_t(1 + i_t)^{n-t} \qquad (\text{ 3.3 })$$

If we assume a uniform series of cash flows with uniform discount rate, say i, over its life, then we can calculate AW by

either

$$A = (A|P\ i\%,\ n)\left[\sum_{t=0}^{n} A_t(1 + i_t)^{-t}\right] \qquad (\text{ 3.4 })$$

or

$$A = (A|F\ i\%,\ n)\left[\sum_{t=0}^{n} A_t(1 + i_t)^{n-t}\right] \qquad (\text{ 3.5 })$$

Please note that the NPV, NFV and AW measures implicitly assume that the intermediate cash flows can be reinvested at the rate of cost of capital.

<u>Example 3.5</u>

Three projects have been suggested to a corporation. The

after-tax cash flows for each project are tabulated below. If corporation's cost of capital is 12%, rank them in order of NPV.

Table 3.4 Cash Flow Profiles

Period	Proj. A	Proj. B	Proj. C
0	$-10,000	$-30,000	$-18,000
1	2,800	6,000	6,500
2	3,000	10,000	6,500
3	4,000	12,000	6,500
4	4,000	16,000	6,500

Compare the projects using Total Cash Flow approach,

$$NPV_A = -10,000 + 2,800(1.12)^{-1} + 3,000(1.12)^{-2} + 4,000(1.12)^{-3} + 4,000(1.12)^{-4}$$
$$= \$280.77$$

$$NPV_B = -30,000 + 6,000(1.12)^{-1} + 10,000(1.12)^{-2} + 12,000(1.12)^{-3} + 16,000(1.12)^{-4}$$
$$= \$2,038.73$$

$$NPV_C = -18,000 + 6,500(1.12)^{-1} + 6,500(1.12)^{-2} + 6,500(1.12)^{-3} + 6,500(1.12)^{-4}$$
$$= \$1,742.77$$

The projects would be ranked as follows: B, C, A

3.1.2.3 Internal Rate of Return (IRR) Method

By definition, the Internal Rate of Return (IRR) is the rate which equates the present value of cash inflows with the present value of cash outflows.

Letting i_r denote the IRR, the following equation should be

satisfied:

$$0 = \sum_{t=0}^{n} A_t (1 + i_I)^{-t} \qquad (3.6)$$

The assumption underlying the IRR is that the cash flows over the project life can be reinvested at the rate of the IRR. When the IRR is greater than the cost of capital, the project is acceptable.

<u>Example 3.6</u>

A project has an after-tax cost of \$10,000, and will result in after-tax cash inflows of \$3,000 in year 1, \$5,000 in year 2 **and \$6,000 in year 3. We are using the IRR measure to determine if the project is acceptable, and assuming that the cost of capital is 15%.**

Solution:

The i_I is determined by the following equation:

$$NPV = -10,000 + 3,000(1 + i_I)^{-1} + 5,000(1 + i_I)^{-2} + 6,000(1 + i_I)^{-3} = 0$$

Letting $i_I = 17\%$, $NPV = -38$

Letting $i_I = 16\%$, $NPV = 146$

So, the actual IRR is between 17% and 16%. We may obtain it by using linear interpolation:

$$i_I = 17\% - (1\%)\left[\frac{0 - (-38)}{146 - (-38)}\right] = 16.79\%$$

Since 16.79% > 15% , we shall accept the project.

Example 3.7

Referring to Table 3.4 of Example 3.5, what will be the rank order of the alternatives using the IRR measure, assuming that the cost of capital is 15% ?

Solution: we will use the incremental Cash Flow approach to solve the problem. The incremental cash flow profiles are as follows:

Table 3.5 Cash Flow & Incremental Cash FLow Profiles

Period	Proj. A	Proj. B	Proj. C	Proj.(C-A)	Proj.(B-C)
0	$-10,000	$-30,000	$-18,000	$-8,000	$-12,000
1	2,800	6,000	6,500	3,700	500
2	3,000	10,000	6,500	3,500	3,500
3	4,000	12,000	6,500	2,500	5,500
4	4,000	16,000	6,500	2,500	9,500

Between Project A and 'Do Nothing',

$$NPV_{A-0} = -10,000 + 2,800(1 + i_I)^{-1} + 3,000(1 + i_I)^{-2} + 4,000(1 + i_I)^{-3} + 4,000(1 + i_I)^{-4}$$

Since $NPV_{A-0}(13\%) = 52.79$ and $NPV_{A-0}(14\%) = -167.25$, the actual IRR can be interpolated by:

$$i_I = 13\% + (1\%)\left(\frac{52.79}{52.79 - (-167.25)}\right) = 13.24\%$$

13.24% is less than cost of capital, so project A is worse than 'Do nothing'.

Between Project C and 'Do Nothing',

$$NPV_{C-0} = -18,000 + 6,500(1 + i_I)^{-1} + 6,500(1 + i_I)^{-2}$$

$$+ 6,500(1 + i_I)^{-3} + 6,500(1 + i_I)^{-4}$$

Since $NPV_{C-0}(16\%) = 188.17$ and $NPV_{C-0}(17\%) = -168.97$, the actual IRR can be interpolated by:

$$i_I = 16\% + (1\%)\left(\frac{188.17}{188.17 - (-168.97)}\right) = 16.53\%$$

16.53% is greater than cost of capital, so project C is better than 'Do nothing'.

Between Project B and C,

$$NPV_{B-C} = -12,000 + 500(1 + i_I)^{-1} + 3,500(1 + i_I)^{-2}$$
$$+ 5,500(1 + i_I)^{-3} + 9,500(1 + i_I)^{-4}$$

Since $NPV_{B-C}(12\%) = 295.96$ nd $NPV_{B-C}(13\%) = -63.16$, the actual IRR can be interpolated by:

$$i_I = 12\% + (1\%)\left(\frac{295.96}{295.96 - (-63.16)}\right) = 12.82\%$$

12.82% is less than cost of capital, so project C is worse than project B.

The rank order among projects should be: B, C, A.

The major drawback to the use of IRR method is the multiple roots in some complex cash flow profiles. It is difficult to **interpret** the polynomial roots. For additional discussions of this subject, interested reader may refer to Bussey [4] and Clark [5].

3.1.2.4 External Rate of Return (ERR) Method

The external rate of return (ERR) method consists of the determination of the rate, i_E, which satisfies the following equation:

$$\sum_{t=0}^{n} R_t(1 + r_t)^{n-t} = \sum_{t=0}^{n} C_t(1 + i_E)^{n-t} \qquad (3.7)$$

where r_t : the cost of capital at period t

R_t : the cash inflow (revenue) at t

C_t : the cash outflow (cost) at t

Of ERR method, we assume that the positive net cash flows can always be reinvested at the cost of capital. ERR method offers a distinct advantage over IRR method, in that there is a unique value of i_E which satisfies the equation. When i_E is greater than the given cost of capital, the alternative should be chosen.

Example 3.8

Referring to Ex. 3.6, determine the project's acceptability by **using the ERR measure, assuming that the cost of capital is** 15%.

Solution: The ERR, i_E can be obtained from the following,

$$10{,}000(1+i_E)^3 = 3{,}000(1+15\%)^2 + 5{,}000(1+15\%)^1 + 6{,}000$$

We can get that $i_E = 16.2\%$ > cost of capital.

Therefore, we shall accept the project.

Example 3.9

Referring to Ex. 3.7, rank the alternatives using ERR **measures, assuming that the cost of capital is 14%.**

Solution: We will use the incremental approach for the ERR measures. Between Project A and 'Do Nothing':

$$10,000(1 + i_E)^4 = 2,800(1+14\%)^3 + 3,000(1+14\%)^2 + 4,000(1+14\%) + 4,000 = 16,607.12$$

Solving it, we get $i_E = 13.52\%$ < cost of capital.

Since i_E is less than cost of capital, alternative A is worse than 'Do Nothing'.

Between Project C and 'Do Nothing':

$$18,000(1 + i_E)^4 = 6,500(1+14\%)^3 + 6,500(1+14\%)^2 + 6,500(1+14\%) + 6,500 = 31,978.44$$

Solving it, we get $i_E = 15.46\%$ > cost of capital.

So, alternative C is better than 'Do Nothing'.

Between Project B and C:

$$12,000(1 + i_E)^4 = 500(1+14\%)^3 + 3,500(1+14\%)^2 + 5,500(1+14\%) + 9,500$$

$= 21,059.37$

Solving it, we get $i_E = 15.09\% >$ cost of capital. So, alternative B is better than C. The ranking order should be B, C, A.

When comparing alternatives by different measures, we can expect disparate order of preference. The main reason for the **inconsistent results may be the inconsistent assumptions** underlying each of the measures. In practice, IRR method is more popular than ERR method.

3.1.2.5 Profitability Index (PI) **Method.**

The profitability index (PI) measure, also called benefit-cost ratio, is the ratio of the net present value of after-tax cash inflows to outflows. When the ratio of a certain project is greater than 1, we may conclude that it has an expected yield greater than the cost of capital. PI can be expressed as follows:

$$PI = \frac{\sum_{t=0}^{n} R_t (1 + i)^{-t}}{\sum_{t=0}^{n} C_t (1 + i)^{-t}} \qquad (3.8)$$

where R_t is the cash inflow (revenue) at t,

C_t is the cash outflow (cost) at t

and i is the cost of capital.

The PI is a measure of a project's profitability per dollar of investment. The assumption that underlies PI is that the cash flows are always reinvested at the cost od capital. If projects are ranked by PI, we must be cautious about the size of the projects. For example, an investment in a typewriter may be better than an investment in a computer according to the PI method, which chooses between two alternatives which are not suitable for comparison.

<u>Example 3.10</u>

Referring to Ex. 3.6, we will determine the acceptability of the project using the PI method, assuming the cost of capital is 15%.

$$PI = \frac{3,000(1.15)^{-1} + 5,000(1.15)^{-2} + 6,000(1.15)^{-3}}{10,000}$$

$$= 1.033 > 1$$

So, we shall accept the project.

<u>Example 3.11</u>

Referring to Ex. 3.7, rank the alternatives in terms of the PI measure, assuming that the cost of capital to the firm is 14%.

Solution: We shall use incremental cash flow approach for this problem.

Between project A and 'Do Nothing',

$$PI = \frac{2800(1.14)^{-1} + 3000(1.14)^{-2} + 4000(1.14)^{-3} + 4000(1.14)^{-4}}{10000}$$

$$= 0.98 < 1$$

So, project A is worse than 'Do Nothing'.

Between project C and 'Do Nothing',

$$PI = \frac{6500(1.14)^{-1} + 6500(1.14)^{-2} + 6500(1.14)^{-3} + 6500(1.14)^{-4}}{18000}$$

$$= 1.052 > 1$$

So, project C is better than 'Do Nothing'.

Between project B and C,

$$PI = \frac{500(1.14)^{-1} + 3500(1.14)^{-2} + 5500(1.14)^{-3} + 9500(1.14)^{-4}}{12000}$$

$$= 1.039 > 1$$

So, project B is better than C.

The order of preference should be: B, C, A.

3.1.3 Discussions -- Conflicts in Ranking Using Different Methods

Conflicts in ranking mutually exclusive projects may result from the difference in the original investment, disparate cash flows,or a lack of comparability due to unequal lives. In this section, we focus our discussion on conflicts of the following typical methods: NPV, IRR and PI. Specifically, conflicts among these measures could arise when:

1) a size disparity exists between the cash outflows,

2) a time disparity exists between the cash inflows and

3) a disparity exists in the useful lives.

3.1.3.1 Size Disparity

Size disparity exists between (or among) projects when they require different magnitudes of discounted outflows. The ranking order by NPV, IRR and PI may be inconsistent under this circumstance. The main reason is that NPV measure the absolute excess cash outflows, whereas PI measures relative profitability, and IRR measures the compounded rate of return which equates the discounted cash inflows and outflows. Let us illustrate the problem and the solution **with examples.**

Example 3.12

A firm, having a 12% cost of capital, is evaluating two mutually exclusive projects, X and Y, which have the following characteristics:

	Proj. X	Proj. Y	(X - Y)
Investment	$-500,000	$-100,000	$-400,000
Cash inflow/year	150,000	40,000	110,000
Useful life	10 yr	10 yr	

Determine the order of preference by NPV, IRR and PI methods.

Solution 1: (total cash flow approach)

The results of these measures are summarized by the following table:

	Proj. X	Proj. Y	Rank X	Rank Y
Cash inflows (discounted at 12%)	$847,533.45	$226,008.92		
Cash outflows	500,000.00	100,000.00		
NPV	$347,533.45	$126,008.92	1	2
IRR	27.32%	38.45%	2	1
PI	1.695	2.26	2	1

To avoid the inconsistency in ranking, Clark suggested us to use incremental cash flow approach to **resolve** these conflicts.

Solution 2: (incremental cash flow approach)

By using the incremental cash flows, the results are summarized as follows:

	(X - Y)	Rank X	Rank Y
Cash inflow/year (for 10 years)	$110,000.00		
Investment	$400,000.00		
NPV (at 12%)	$221,524.53 > 0	1	2
IRR	27.32% > 12%	1	2
PI	1.695 > 1	1	2

By incremental aproach, we can see that the ranking are justified.

3.1.3.2 Time Disparity

The time disparity exists when there are differences with

respect to the sequence of time of cash inflows. In this case, NPV and PI will always produce consistent ranking. Conflicts arise between NPV (or PI) and IRR because of the different implicit assumptions about the rate of return that can be earned on intermediate cash inflows. NPV (or PI) method assumes the intermediate cash inflows can be reinvested at the hurdle rate (cost of capital), while IRR method assumes at IRR.

The conflicts of the reinvestment assumption can be solved by incoporating a true reinvestment rate **into the three measures**. This is accomplished by calculating the terminal value (TV) of the project, given that the intermediate cash inflows can be reinvested at a specified rate. TV for a given project is determined by the following equation:

$$TV = \sum_{t=1}^{n} R_t (1 + k)^{n-t} \qquad (3.9)$$

where R_t : cash inflow at the end of period t
k : the reinvestment rate
n : useful life

The modified NPV method (denoted by NPV^*) would employ:

$$NPV^* = \frac{TV}{(1 + i)^n} - A_o \qquad (3.10)$$

where i : cost of capital
A_o: net discounted cash outflow

The modified IRR method (denoted by IRR*) would be implemented using the following equation:

$$\frac{TV}{(1 + i^*)^n} - A_o = 0 \qquad (3.11)$$

where i^* : the true internal rate of return

Let us illustrate the time disparity problem by the following example.

Example 3.13 [5]

A firm whose cost of capital is 10% is considering two mutually exclusive projects, A and B, whose characteristics are as follows:

		Proj. A	Proj. B
Investment		-$70,000	-$70,000
Cash FLows	yr. 1	10,000	50,000
	yr. 2	20,000	40,000
	yr. 3	30,000	20,000
	yr. 4	45,000	10,000
	yr. 5	60,000	10,000

Determine the order of preference by NPV, PI and IRR measures.

Solution 1: (without justifying the reinvestment assumption)

The NPV, PI and IRR measures are calculated and summarized in the following table.

	Proj. A	Proj. B	Rank A	Rank B
Discounted Cash inflows at 10%	116,150.16	$106,578.03		
NPV	46,150.16	36,578.03	1	2
PI	1.659	1.523	1	2
IRR	27.2%	37.55%	2	1

We can see that the IRR measure favors the project having high cash inflows early in the project life. To solve the conflict , we need to know the reinvestment rate for cash inflows. Assuming that the reinvestment rate is 20%, then what will be the rank order by NPV, PI and IRR methods?

Solution 2: (justifying the reinvestment assumption)

For project A,

$$TV_A = 10{,}000(1.2)^4 + 20{,}000(1.2)^3 + 30{,}000(1.2)^2 + 45{,}000(1.2) + 60{,}000 = 212{,}496.00$$

$$NPV_A^* = \frac{212{,}496}{(1.10)^5} - 70{,}000 = 61{,}943.30$$

$$IRR_A^*: \quad \frac{212{,}496}{(1 + i_A^*)^5} - 70{,}000 = 0$$

Solving the above equation, we get $i_A^* = 24.87\%$

For project B,

$$TV_B = 50{,}000(1.2)^4 + 40{,}000(1.2)^3 + 20{,}000(1.2)^2 + 10{,}000(1.2) + 10{,}000 = 223{,}600.00$$

$$NPV_B^* = \frac{223{,}600}{(1.10)^5} - 70{,}000 = 68{,}838.00$$

$$IRR_B^*: \quad \frac{223{,}600}{(1 + i_B^*)^5} - 70{,}000 = 0$$

Solving the above equation, we get $i_B^* = 26.2\%$

Therefore, we prefer project B to A. The order of preference measured by different methods **is** now consistent.

More generally, when the reinvestment rate is changing over time, the TV calculation should be as follows:

$$TV = \sum_{t=0}^{n} R_t \left[\prod_{j=t+1}^{n} (1 + k_j) \right] \qquad (3.12)$$

where k_j : the reinvestment rate in period j

When discount rate changes over project life, the NPV^* is as follows:

$$NPV^* = \frac{TV}{\prod_{t=1}^{n} (1 + i_t)} - A_o \qquad (3.13)$$

where i_t is the cost of capital for period t

3.1.3.3 Unequal Useful Lives

To solve the conflicts due to unequal project lives, we may reasonably assume that each investment will be replaced at the end of its life with another **investment of the same** profitability. The project with shorter life, in this sense, can repeat **its** investment and extend its life to a certain horizon where projects can be compared.

<u>Example</u> <u>3.14</u> [5]

Consider a firm (whose cost of capital is 10%) which is evaluating two projects having the following characteristics:

t	Proj. X	Proj. Y
0	-100	-100
1	+120	0
2		0
3		0
4		+174.9

Rank the projects by NPV, PI and IRR measures.

Solution 1: (without justifying the useful lives)

The NPVs, PIs and IRRs are as follows:

	Proj. X	Proj. Y	Rank X	Rank Y
NPV (at 10%)	9.09	19.46	2	1
PI (at 10%)	1.0909	1.1946	2	1
IRR	20%	15%	1	2

This approach is not proper since it does not consider the projects' unequal lives. The ranking orders are inconsistent.

Solution 2: (justifying the useful lives)

We will assume the replacement of the project at the end of its life. The cash flow profile of project X should be revised as follows :

time	1	2	3	4	Net CF	PV factor	discounted CF
0	-100				-100	1.00000	-100.00
1	+120	-100			20	.90909	18.18
2		+120	-100		20	.82645	16.53
3			+120	-100	20	.75131	15.03
4				+120	+120	.68301	81.96
							31.07

The NPV, PI and IRR measures are revised as follows:

	Proj. X	Proj. Y	Rank X	Rank Y
NPV (at 10%)	31.07	19.46	1	2
PI (at 10%)	1.3170	1.1946	1	2
IRR	20%	15%	1	2

Now the ordering is consistent. Project X is better than Y.

3.2 Project Evaluation and Ranking --- Under Risk

Townsend [40] defines the term 'risk' as any situation in which we do not know the outcome of events with certainty, but we do know the following:

(1) the number of alternative possible outcomes, and

(2) the 'value' of each outcome, and

(3) the probability of occurrence of each outcome.

It is a class of uncertainty in which specific data is available to us. The statistical analysis is used in evaluating projects under risk conditions.

3.2.1 Expectation-Variance (E-V) Principle

If **X** is a discrete random variable that is defined for a finite number of values, and **P**(x) denotes the probability of a particular value, x, occurring, then the expected value of the random variable, **X**, is defined by:

$$E(\mathbf{X}) = \sum_{all\ x} x\ \mathbf{P}(x) \qquad (3.14)$$

The variance of the random variable is defined by

$$Var(X) = \sigma_x^2 = \sum_{all\ x} [x - E(X)]^2 P(x) \qquad (3.15)$$

where σ_x: the standard deviation of X

$$\text{or, } Var(X) = E(X^2) - E(X)^2 \qquad (3.16)$$

The expected value can be interpreted as the average value of outcomes that we are expecting, and the variance as the variation (the degree of uncertainty) of our expectation. **Higher E(X) means greater gain, while larger Var(X) indicates** greater uncertainty. People tend to prefer high expected value and low variance. There are trade-off between E(X) and Var(X). To determine the preference order by measuring E(X) and Var(X), the decision maker's utility valuation between their trade-off should be known. Let us illustrate the method by the following example.

Example 3.15

Three alternatives are to be evaluated. Their possible outcomes and associated probabilities are shown below:

Table 3.6 Possible outcomes table

Cost of Capital	10%	20%	30%
Probability	.1	.3	.6
NPV (Alt. A)	15,136	11,962	9,742
NPV (Alt. B)	16,536	10,934	7,049
NPV (Alt. C)	18,397	10,840	5,679

Evaluate the alternatives using E-V principle.

Solution:

The expected values of each alternative are calculated as follows:

$$E(A) = 15{,}163(.1) + 11{,}962(.3) + 9{,}742(.6)$$
$$= 10{,}950.1$$

$$E(B) = 16{,}536(.1) + 10{,}934(.3) + 7{,}049(.6)$$
$$= 9{,}163.2$$

$$E(C) = 18{,}397(.1) + 10{,}840(.3) + 5{,}679(.6)$$
$$= 8{,}499.1$$

The variances of each alternative are calculated as follows:

$$Var(A) = (15163 - 10950.1)^2(.1) + (11962 - 10950.1)^2(.3) + (9742 - 10950.1)^2(.6)$$
$$= 2{,}082{,}035.12 \qquad \sigma_A = 1{,}442.9$$

$$Var(B) = (16536 - 9163.2)^2(.1) + (10934 - 9163.2)^2(.3) + (7049 - 9163.2)^2(.6)$$
$$= 9{,}058{,}442.76 \qquad \sigma_B = 3{,}009.7$$

$$Var(C) = (18397 - 8499.1)^2(.1) + (10840 - 8499.1)^2(.3) + (5679 - 9499.1)^2(.6)$$
$$= 10{,}212{,}564.69 \qquad \sigma_C = 4{,}026.5$$

In this example, $E(A) > E(B) > E(C)$ and $\sigma_A < \sigma_B < \sigma_C$, so, Project A dominated B, and B dominates C. The order of preference should be A, B, C.

Normally, the E-V analysis will yield nondominated results.

In this case, the decision makerS should determine their valuation of preference. An aggressive Decision maker may prefer high expectation projects while a conservative decision maker may stick to low variation projects.

3.2.2 Risk Analysis

When probability distribution function of cash flow outcomes is known, we are able to analyze the expected value and variance associated with the outcomes. Suppose that C_j's are random variables with expected value $E(C_j)$ and variance $Var(C_j)$. Consider the following present worth relation:

$$PW = \sum_{t=0}^{n} C_j(1 + i)^{-j} \qquad (\ 2.8,\ \text{recalled}\ )$$

The expected value of PW is the sum of random variables, a linear combination, which is given by:

$$E(PW) = \sum_{j=0}^{n} E[C_j(1 + i)^{-j}] = \sum_{j=0}^{n} E(C_j)(1+i)^{-j} \qquad (\ 3.17\ )$$

The variance of such linear combination is given by:

$$Var(PW) = \sum_{j=0}^{n} Var(C_j)(1+i)^{-2j} + 2\sum_{j=0}^{n-1}\sum_{k=j+1}^{n} Cov(C_j,C_k)(1+i)^{-(j+k)}$$

$$(\ 3.18\ )$$

where $Cov(C_j, C_k)$: covariance of C_j and C_k

When the random events are mutually independent from each other, the covariance should be equal to zero. Thus, Eq.

(3.18) can be reduced as follows:

$$Var(PW) = \sum_{j=o}^{n} Var(C_j)(1+i)^{-2j} \qquad (3.19)$$

With known E(PW) and Var(PW), and assuming a normlly distributed PW (the assumption is acceptable due to the *central limit theorem*), we may test out a certain hypothesis with respect to each alternative. The higher the probability or confidence interval associated with the alternative, the better order of preference may be assigned to it.

Example 3.16

An investment of $8,000 is estimated to have useful life of 5 years. The return of each period t, denoted R_t, is a random variable with probabilities and outcomes depicted as follows:

R_t (t=1,2,3,4,5)	Probability
1,000	.1
1,500	.2
2,000	.1
3,500	.2
4,000	.4

By using the 12% cost of capital, decide whether the investment is profitable. Assuming that R_t and R_j ($t \neq j$) are independent with each other and the decision maker requests a 95% possibility of positive net cash flow.

Solution:

The expected value and variance of R_t (t = 1,2,3,4,5) are:

$E(R_t) = 1000(.1) + 1500(.2) + 2000(.1) + 3500(.2) + 4000(.4)$

$= 2,900$

$$Var(R_t) = E(R_t^2) - [E(R_t)]^2$$

$$= [1000^2(.1) + 1500^2(.2) + 2000^2(.1) + 3500^2(.2) + 4000^2(.4)] - 2900^2$$

$$= 1,390,000 \qquad \sigma_{R_t} = 1,178.98$$

Therefore,

$$E(PW) = \sum_{t=1}^{5} E(R_t)(1 + .12)^{-t} - 8000 = 2900[(1.12)^{-1} + (1.12)^{-2} + (1.12)^{-3} + (1.12)^{-4} + (1.12)^{-5}] - 8000$$

$$= 2,453.85$$

$$Var(PW) = 139000[(1.12)^{-2} + (1.12)^{-4} + (1.12)^{-6} + (1.12)^{-8} + (1.12)^{-10}]$$

$$= 3,704,627.36 \qquad \sigma_{PW} = 1,924.74$$

Now we are trying to find the probability of positive net cash flow , which can be obtained by standarization of PW to the standard Normal distribution (whose mean is 0 and variance is 1).

$$P(PW \geq 0) = P(Z \geq \frac{0 - E(PW)}{\sigma_{PW}}) = P(Z \geq \frac{0 - 2453.85}{1924.74})$$

$$= P(Z \geq -1.275)$$

From Standard Normal table, $P(Z \geq -1.28) = 0.8997$ and $P(Z \geq -1.27) = 0.8980$. Therefore, we can get $P(Z \geq -1.275)$ by linear interpolation, where $P(Z \geq -1.275) = 0.89885$.

Since a 95% guarantee is required, we shall not accept this project. project.

The example illustrates the calculation to evaluate individual project under the risk condition. When more alternatives are to be evaluated, the calculations are the same. The order of preference is then based on the order of probabilities -- the larger, the better.

3.2.3 Most Probable Future Principle

The most probable future principle is to consider the state having a probability of occurrence considerably greater than any other. The decision is thereby reduced to a decision under assumed certainty. Then, an alternative is chosen that maximize (or minimize) the measure of effectiveness. Let us see the following example.

Example 3.17

From Table 3.8 in Ex. 3.15, the most probable future of each alternative is :

A : 9,742

B : 7,049

C : 5,679

Since our objective is to maximize present worth, the order of preference should be: A, B, C.

It should be noted that the principle is proper only when one onoutcome has a significantly higher probability than any than any other, and the values of the outcomes do not differ

significantly.

3.2.4 Aspiration Level principle

In most cases, the decision makers would set a certain aspiration level before evaluating alternatives. An interpretation of this philosophy in terms of a decision under risk is to select an alternative that maximizes the probability of achieving the desired aspiration level. Typical aspiration level, objective might be to choose alternatives that maximize the probability of (1) a certain IRR level, (2) a certain cost level or (3) a certain profit level. Let us illustrate the principle by the following example.

<u>Example 3.18</u>

In terms of Ex. 3.15, suppose the objective is to choose alternative which has the highest probability of obtaining present worth greater than 8,000. Let us symbolize it as $P(PW \geq 8000)$. We can see from Table 3.8 that,

for A , $P(PW \geq 8000) = .1 + .3 + .6 = 1.0$

for B , $P(PW \geq 8000) = .1 + .3 = .4$

for C , $P(PW \geq 8000) = .1 + .3 = .4$

Of course, A is the best alternative.

3.2.5 Certainty Equivalent (CE) Method

The Certainty Equivalent (CE) method permits adjustment for

risk by incorporating the manager's utility preference for risk versus return directly into capital investment processes. This **method is especially useful when management perceives different levels of risk associated with the estimated annual cash flow over the life of a project.**

It is reasonable to assume that the estimates of cash flows during the early period in project's life are likely to be more accurate than those corresponding to latter years. When certainty equivalent method is used, the estimated annual cash flows are multiplied by a Certainty Equivalent Coefficient (**CEC**), denoted as α. The **CEC** reflects management's degree of **aversion to perceived risk. The CEC's are tanged in value from 0 to 1. Higher values of CEC indicate lower penalities assigned by the management.**

An interpretation of cost of capital is that it reflects the risk-premium rate and the risk-free rate of return to compensate for the investment. The certainty equivalent method is designed to compensate for risk in its certainty. Therefore, the risk-free rate of return is the appropriate discount rate when this method is used. The certainty equivalent value is defined as follows:

$$NPV_{CE} = \sum_{t=o}^{n} \frac{\alpha_t \bar{R}_t}{(1 + i)^t} \qquad (3.20)$$

where NPV_{CE} : expected certainty equivalent value

$\bar{R}_t$: expected cash inflow in period t

α_t : certainty equivalent factor in period t

i : risk-free rate

n : project's useful life

More generally, when the risk-free rate does not remain constant over time, the equation should be rewritten as follows:

$$NPV_{CE} = \sum_{t=o}^{n} \frac{\alpha_t \bar{R}_t}{\prod_{k=1}^{n} (1 + i_k)^t} \qquad (3.21)$$

where i_k : the risk-free rate in year k

A method for developing certainty equivalent cofficient value is demonstrated in the following example.

<u>Example 3.19</u> [5]

A corporation has a cost of capital of 10% and the risk-free rate of return at 5%. A project falling within the firm's normal risk posture is being evaluated. It has a 1-year life and an expected cash flow of $1,000 to be received at the end of year 1. Determine the **CEC** of the project.

Solution: Let us first determine the PV of the project.

$$PV = \frac{1000}{(1 + .1)} = 909.09$$

The corresponding **CEC** may be found by the following

equation:

$$\frac{\alpha_1(1000)}{(1 + .05)} = 909.09$$

We may get that $\alpha_1 = .95454$

The CEC value in a single period is easily found as shown in the above example. However, since most projects have lives that extend over several periods, it is necessary to incorporate management's risk preference on a multi-period basis. One procedure for ascertaining CEC values is to undertake the historical review of projects' performance. Projects are divided into categories, then within each category, on a year-by-year basis, the measure of risk and return are determined. The result is a probability distribution of cash flows by year of project life, from which the coefficient of variation may be obtained. The CEC for each year and each category of project would then be assigned according to the magnitude of the coefficient of variation weighted by managers' preference for risk aversion. An example of the result is shown in Table 3.7 for 4-year period projects based on the utility preference for the firm at a point in time.

The procedure for using certainty equivalent coefficients is demonstrated in Example 3.20.

Table 3.7 Certainty Equivalent Factors for Different Investment Groups ([5], simplified)

Investment Grouping		Coefficient of Variaton, ν^*	Certainty Equivalent Coefficient (α)			
			yr 1	yr 2	yr 3	yr 4
Replacement	— category 1	$\nu < .1$	.95	.92	.89	.85
	— category 2	$.1 \le \nu \le .25$	.90	.86	.82	.77
	— category 3	$\nu > .25$	.84	.79	.74	.68
New Investment	– category 1	$\nu < .1$	.92	.88	.85	.80
	– category 2	$.1 \le \nu \le .25$	.86	.82	.78	.73
	– category 3	$\nu > .25$	.80	.75	.70	.64
R&D development	– category 1	$\nu \le .2$	.82	.76	.70	.60
	category 2	$\nu > .2$	.70	.60	.50	0

* $\nu = \dfrac{E(X)}{\sigma_X}$, a dimensionless quantity which measure the amount of variability relative to mean value

Example 3.20 [5]

A new investment project has expected returns and standard deviations during its 4-year life as follows:

Year	Expected Return	Standard Deviation	Coefficient of Variation
1	$1,000	$200	0.20
2	1,200	216	0.18
3	1,200	168	0.14
4	1,800	144	0.08

The cost is $3,000 in the very beginning and the risk-free rate of return is 6%. Determine the certainty equivalent value

according to the CEC factors listed in Table 3.7.

Solution: The calculations of NPV_{CE} are summarized in the following table :

Time	$\bar{R}_t$	α_t	$\alpha_t \bar{R}_t$	PV factor (at 6%)	Discounted $\alpha_t \bar{R}_t$
0	-3,000	1.0	-3,000	1.0	-3,000
1	1,000	.86	860	.943	811
2	1,200	.82	984	.890	876
3	1,200	.78	936	.840	786
4	1,800	.80	1,440	.792	1,140
				NPV_{CE} =	\$ 613

Since the project yields a positive NPV_{CE}, it is therefore **representative of an acceptable candidate.**

There are some weak points for the CE **method. One** is that the determination of α's required experienced management which can clearly define their utility preference. Another is the predetermined categorization of investment groups, which may not be able to include all possible investment projects. The **last** is the predetermined time horizon in each category of investment, which implies that the decision makers are completely **disregarding** the cash flows other than the time horizon.

3.2.6 Risk-Adjusted Discount Rate (RAR) Method

The rationale underlying the use of Risk-Adjusted discount Rate (RAR) method is that projects having greater variability in the **probable** . distribution of their returns should have these return discounted at a higher rate than those having less variability or risk. The risk-adjusted rate is represented by the following equation:

$$r' = i + u + a \qquad (3.22)$$

where r' : the risk-adjusted discount rate

i : risk-free rate

u : adjustment for the firm's normal risk

a : adjustment for above (or below) the firm's normal risk

It should be noted that the sum of i and u is simply the firm's cost of capital. We should **also note that** the amount of risk-adjustment is based on the management's utility preference for risk aversion. Therefore, the risk adjustment reflects the management's perception of risk associated with projects. The following equation is used to determine the expected present value when employing risk-adjusted discount rate:

$$NPV_{RAR} = \sum_{t=o}^{n} \frac{\bar{R}_t}{(1 + r')^t} \qquad (3.23)$$

where NPV_{RAR} : expected risk-adjusted net present value

$\bar{R}_t$: expected value of cash flow in period t

r′ : risk-adjusted discount rate

n : project life

Table 3.8 [5] is an example table indicating a firm's required return for various investment categories. We **should note that**, unlike the CE method, the **RAR method** applies the same discount rate to the project throughout its useful life. It means the rate of risk is assumed the same for the entire life of the project.

Table 3.8 Return Required for Various Investment Group [5]

Investment Grouping		Required Return
Replacement	– category 1	Cost of capital
	category 2	Cost of capital + 3%
	category 3	Cost of capital + 6%
New Investment	– category 1	Cost of capital + 5%
	category 2	Cost of capital + 8%
	category 3	Cost of capital + 15%
R&D Investment	– category 1	Cost of capital + 10%
	category 2	Cost of capital + 20%

Let us illustrate the application by the following example.

Example 3.21 [5]

A firm is considering a "Replacement - category 2" project,

which has the cash flows shown by the following distribution:

Original Cost		Cash Flows			
		Year 1 - 5		Year 6 - 10	
Prob.	Amount	Prob.	Amount	Prob.	Amount
.3	$13,000	.2	$2,000	.2	$2,600
.4	14,000	.4	2,400	.6	3,200
.3	15,000	.3	2,800	.1	3,400
		.1	3,400	.1	3,600

Assuming that the firm's cost of capital is 11%, determine the risk-adjusted net present value.

Solution : The expected cost is obtained by the following:

$\overline{C} = 13000(.3) + 14000(.4) + 15000(.3) = \$14,000$

The $\bar{R}_t$ for each year is calculated as follows:

for $t = 1, 2, 3, 4, 5$

$R_t = 2000(.2) + 2400(.4) + 2800(.3) + 3400(.1)$

$= \$2,540$

for $t = 6, 7, 8, 9, 10$

$R_t = 2600(.2) + 3200(.6) + 3400(.1) + 3600(.1)$

$= \$3,140$

The risk-adjusted NPV is determined as follows:

$$NPV_{RAR} = -14,000 + \sum_{t=1}^{5} \frac{2,540}{(1.14)^t} + \sum_{t=6}^{10} \frac{3,140}{(1.14)^t}$$

$$= \$319$$

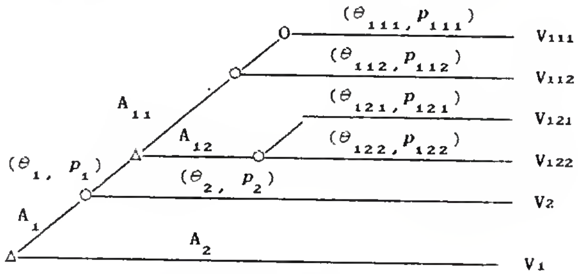
Since it yields a positive value, the project represents an acceptable candidate.

RAR method has been criticized for the following reasons:
1) The method uses a constant rate for a certain category of projects which does not examine the riskiness associated with each project or the changing in riskness over projects' lives.
2) The use of high discount rate results in an exponential growth (or decay) of the values of cash flows as a function of time. The difference between the NPV discounted at risk-free rate and the NPV discounted at risk-adjusted rate increases exponentially with the passage of time.

3.2.7 Decision Trees

The decision trees technique is useful when the results of a project are conditioned on its previous outcomes. The technique solves the sequential decisions in stochastic decision-trees by comparing their expectation criteria. Due to the development of computer techniques, Jones [19] published an article "Decision analysis using spreadsheet", which shows that a spreadsheet package, Lotus 1-2-3, in personal computer

can be used in a surprisingly powerful way to build, solve and perform sensitivity analysis on decision tree. In this section , we will show the construction of a decision tree and series of examples which demonstrate the techniques by using Lotus 1-2-3. A symbolic decision tree diagram is shown on Figure 3.3.



Δ : a decision node
○ : a chance outcome node
(θ, p) : a certain outcome θ with probability p
A : alternative
V : the value associated with a particular branch

Figure 3.3 A Symbolic Desision Tree ([42], revised)

Let us illustrate the decision-tree techniques by demonstrating a series of examples solving a fabricated problem which is stated as follows:

<u>Problem</u>:

The Jay company is a medium-sized manufacturing plant which produces electronic products. During the past few years, the sales have increased sharply, and the company is operating at

almost its full capacity. Facing a fast growing market, the management has proposed two alternatives: (A) to rearrange the existing plant, or (B) to build an additional plant. For alternative (B), management also proposed that if the market demand is sustained at its high level for 2 years, they will then decide whether to expand the plant or not.

Marketing estimates offer the following information:

(1) If an additional plant is built, the market will mainly be occupied by the company. An 80% chance of high demand and a 20% chance of low demand are expected.

(2) If the company rearrange its plant, the increase of output is limited. A 60% chance of high demand and a 40% chance of low demand are expected.

(3) The company rearranges its plant and the high demand is sustained for 2 years: If the company then chooses expansion, a 90% chance of high demand and a 10% chance of low demand from then on are expected; if the company decides against expanding, a 50% chance of high demand and a 50% chance of low demand from then on are expected.

The data associated with each possible outcome are summarized in Table 3.9 .

Table 3.9. Data of the Decision Tree Problem

Discount rate .... 15.0 %
Project life (years) 5

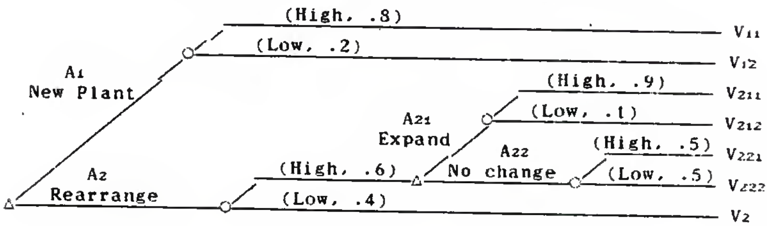
Alternatives & EstimationS	Investment	Probability	Annual Yield
[Alt. 1] -- New Plant:	8000	***	***
.High demand		0.8	3000
.Low demand		0.2	1800
[Alt. 2] -- Rearrange:	3000	***	***
.High demand		0.6	1500
.Low demand		0.4	1000
Subsequent decisions if high demands will sustain for 2 years:			
(Alt. 2-1) -- Expand	4000	***	***
.High demand		0.9	2800
.Low demand		0.1	1800
(Alt. 2-2) -- No Change		***	***
.High demand		0.5	1200
.Low demand		0.5	800

NOTE all the cash flow are in x1000 dollar unit

Example 3.22

Construct the decision tree of the problem stated above.

Solution: The decision tree is as follows:



Example 3.23

Solve the problem by electronic spreadsheet.

Solution: The cash flows profiles are summarized as follows:

Alternative	Demand	Year 1	Year 2	Year 3	Year 4	Year 5	NPV
		Cash Flows					
New Plant	High	-5000	3000	3000	3000	3000	3564.94
	Low	-6200	1800	1800	1800	1800	-1061.04
Rearrange	Low	-2000	1000	1000	1000	1000	854.98
Rearrange & Expand	High	-1500	1500	-1200	2800	2800	2338.93
	Low	-1500	1500	-2200	1800	1800	353.52
Rearrange & No change	High	1500	1500	1200	1200	1200	5186.84
	Low	-1500	1500	800	800	800	1392.68

The solution of the tree is shown by the following figure.

```
              Initial               Subseq.                  Secondary               Subseq.
Yield         Decision     Yield    Demand  Prob.   Yield    Decision    Yield       Demand  Prob.   Yield
-----         --------     ------   ------- -----   ------   --------    -------     ------  -----   ------

                                    +-- High   0.8   3564.94
                                    |                                          [D] : decision node
          +-- New plant  2639.74 --(C)
          |                         |                                          (C) : chance outcome node
          |                         +--- Low   0.2  -1061.04
          |
2639.74 --[D]                                                                  +-- High   0.9   2338.93
          |                                             +-- Expand   2148.39 --(C)
          |                                             |                      +--- Low   0.1    353.52
          |                         +-- High   0.6   3289.76 --[D]
          |                         |                   |                      +-- High   0.5   5186.84
          +-- Rearrange  2315.85 --(C)                  +-No change  3289.76 --(C)
                                    |                                          +--- Low   0.5   1392.68
                                    +--- Low   0.4    854.98
```

Figure 3.4 The analyzed Decision Tree

So, we build a new plant for a better expected net gain.

Example 3.24

The Decision **Maker are not quite sure about** the discount rate. The break-even analysis is requested **against** the fluctuation of discount rate.

Solution: The task is also performed by Lotus 1-2-3. The NPV results associated with changing discount rates are recorded for analysis. The data are summarized in the Table 3.10 . We can see from Table 3.10 that the break-even point is at the rate of 18.45%. When discount rate is less than 18.45%, we shall build a new plant **Otherwise**, we shall rearrange existing plant.

Table 3.10 Break-Even analysis summary (one-way analysis on discount rate)

Discount Rate	NPV New Plant	NPV Rearrange	Decision
0.0%	5800.00	3500.00	↑
2.0%	5269.33	3301.85	
4.0%	4778.51	3118.36	
6.0%	4323.69	2948.12	
8.0%	3901.47	2789.90	New
10.0%	3508.83	2642.59	Plant
12.0%	3143.08	2505.21	
14.0%	2801.85	2376.87	
16.0%	2482.98	2256.80	
18.0%	2184.57	2144.30	
18.4512%	2119.90	2119.90	Break-Even
20.0%	1904.91	2038.73	
22.0%	1642.45	1939.54	
24.0%	1395.80	1846.21	Rearrange
. .	. . .	. . .	↓

Also, by the graph capability of Lotus 1-2-3, we can plot the break-even results in the following graph.

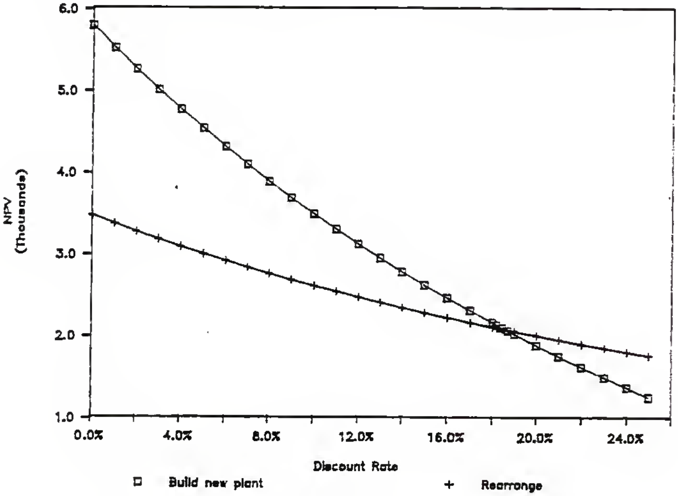


Figure 3.4 Break-Even Analysis (one way on discount rate)

Example 3.25

In this example, a sensitivity analysis is performed against **the probability for high demand when building a new plant.**

Solution: By changing the probabilities and keeping records of the results, we can perform sensitivity analysis similar to the break-even analysis. The results are shown in the following table. We may conclude that when the probability of high demand is less than 0.73, we shall rearrange the

existing plant, otherwise we shall build a new plant.

Table 3.11 One-Way Sensitivity Analysis
(on the probability of New-Plant High Demand)
assuming cost of capital = 15%

Probability	Optimal NPV	Decision
0.0	2315.85	Rearrange
0.1	2315.85	"
0.2	2315.85	"
0.3	2315.85	"
0.4	2315.85	"
0.5	2315.85	"
0.6	2315.85	"
0.7	2315.85	"
0.729985	2315.85	<<either>>
0.8	2639.74	New Plant
0.9	3102.34	"
1.0	3564.94	"

Also, the analysis can be ploted by Lotus 1-2-3, as shown in the following graph.

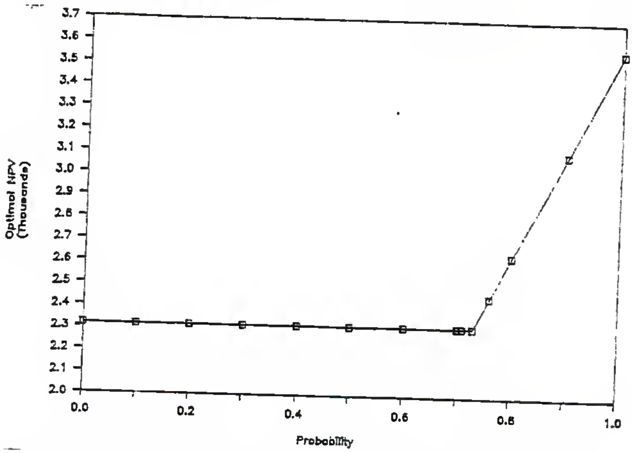


Figure 3.5 Sensitivity Analysis (on probability of high demand)

3.2.7 Monte Carlo Simulation

In section 3.2.2, we discussed the analytical approach to risk analysis by assuming that C_j's are random variables. The approach is somewhat crude because we only considered one random event in a given period. When there are more random events in a period, or the number of alternative values which each variable has is large, then the number of calculations required to obtain the probability distribution of outcomes is **a tremendously effort-consuming task, and thus makes the** analytical risk analysis impossible. Let us explain the effect by a simple example. If a model has n random variables, and each has $m_1, m_2, \ldots, m_n$ possible outcomes. Then the number of possible outcomes, N, is given by:

$$N = m_1 \times m_2 \times \ldots \times m_n \qquad (3.24)$$

Assuming that $n = 4$, and $m_1 = m_2 = m_3 = m_4 = 30$, then the possible outcomes N is given by:

$N = 30^4 = 810{,}000$, which is a quite large number.

Therefore, the stochastic model cannot normally be solved by the analytical approach. Instead, it can be solved by a simulation approach.

Simulation, which tries to imitate a real world system by using a mathematical model, can capture the critical characteristics of the system as it moves through time

encountering random events. In this section, we focus our discussion on a certain simulation model, the Monte Carlo Simulation.

Monte Carlo simulation includes the following processes:

(1) The mathematical model build-up: A model which establish the relationship between the outcome and all the independent variables.

(2) The probability distribution and the cumulative density function (CDF) of each random variable: The probability distribution of each random variable can be obtained by technical analysis, examing historical data or sampling. To evaluate the sampling result for a distribution, we can compare it to a theoretical distribution by performing statistical goodness-of-fit tests. One is the χ^2 goodness-of-fit test [Devore, 08, 522-539], which is proper when the sampling size is large enough. The other is the Kolmogorov-Smirnov test which also is proper for small-sized sampling.

For example, the useful life of project may be a random event with the following probability distribution and CDF:

Useful life	0	1	2	3
Probability	0	.1	.6	.3
CDF	0	.1	.7	1.0

(3) The inverse function of CDF of each random variable: With the inverse function, we are able to simulate the outcomes of a random event by retrieving a random number (between 0 and 1), assuming that it is the value of CDF, and finding its corresponding value of the random variable by the inverse function. **For example,** considering the random event of project's useful life as described above, the inverse CDF of the random variable should be as follows:

Random Number	Useful life
$0 - 0.1$	1
$0.1^{+} - 0.7$	2
$0.7^{+} - 1.0$	3

(4) Generate pseudo random numbers for each random variable and evaluate the outcomes of each random variable: Random numbers can be generated by **random** number table, or by the **generator** in computer. In Lotus, we can generate the random number by invoking the function **@RAND**. The outcomes of the random variables are then obtained by the inverse CDF.

(5) Incorporate all variables into mathematical model: With the simulated outcomes of each random variable, we may calculate the possible result of the model.

(6) Repeat step (4) and (5) numerous times:

(7) Statistical analysis against the model outputs.

Let us illustrate the Monte Carlo Simulation by a simple example which was solved using Lotus 1-2-3.

Example 3.26

The Jay **company** is evaluating an investment project which has uncertainty associated with three important aspects: (1) the Investment cost, (2) the useful life and (3) the annual yield. The probability distribution of the three random variables are shown below:

Cost		Useful life		Annual yield (R)	
value	prob.	value	prob.	value	prob.
5,000	.6	3	.3	1,500	.1
7,000	.4	4	.4	2,000	.2
		5	.3	2,500	.5
				3,000	.2

Determine whether the project is acceptable or not by simulation approach. Assume the cost of capital is 15.0%

Solution:

(1) The mathematical model build-up

The model is expressed as follows:

$$NPV = \sum_{t=1}^{n} \frac{R_t}{(1 + i)^t} - C_0 \qquad (3.25)$$

where R_t : annual yield

C_0 : the initial investment

(2) The probability distribution, the CDF and

(3) The inverse function of CDF of each random variable.

The following table shows the results of these two steps.

Investment cost			
Value	Prob.	CDF	R.No.
4,000	.4	.4	0_+ - .4
6,000	.6	1.0	.4 - 1.0

Useful life			
Value	Prob.	CDF	R.No.
3	.3	.3	0_+ - .3
4	.4	.7	$.3_+$ - .7
5	.3	1.0	.7 - 1.0

Annual yield			
Value	Prob.	CDF	R.No.
1,500	.1	.1	0_+ - .1
2,000	.2	.3	$.1_+$ - .3
2,500	.5	.8	$.3_+$ - .8
3,000	.2	1.0	.8 - 1.0

(4) Generate random numbers, evaluate the outcomes,

(5) Incorporate all variables into the mathematical model and

(6) Repeat step (4) and (5) numerous times.

By using Lotus 1-2-3, the results of steps (4), (5) and (6) are summarized in the Table 3.12 .

Table 3.12 Simulation Results

RUN	Investment R.N	Investment Value	Useful Life R.N	Useful Life Value	Annual Yield R.N	Annual Yield Value	Cash Flows yr 1	yr 2	yr 3	yr 4	yr 5	NPV
1	0.580	6000	0.661	4	0.564	2500	-3500	2500	2500	2500	0	2208.06
2	0.956	6000	0.385	4	0.947	3000	-3000	3000	3000	3000	0	3849.60
3	0.080	4000	0.759	5	0.216	2000	-2000	2000	2000	2000	2000	3709.96
4	0.959	6000	0.767	5	0.752	2500	-3500	2500	2500	2500	2500	3637.45
5	0.977	6000	0.950	5	0.654	2500	-3500	2500	2500	2500	2500	3637.45
6	0.400	4000	0.875	3	0.170	2000	-2000	2000	2000	0	0	1251.42
7	0.701	6000	0.786	5	0.634	2500	-3500	2500	2500	2500	2500	3637.45
8	0.887	4000	0.931	5	0.654	2500	-1500	2500	2500	2500	2500	5637.45
9	0.996	6000	0.019	3	0.853	3000	-3000	3000	3000	0	0	1877.13
10	0.881	6000	0.327	4	0.227	2000	-4000	2000	2000	2000	0	566.45
11	0.771	6000	0.942	5	0.658	2500	-3500	2500	2500	2500	2500	3637.45
12	0.305	4000	0.230	3	0.670	2500	-1500	2500	2500	0	0	2564.27
13	0.273	4000	0.345	4	0.488	2500	-1500	2500	2500	2500	0	4208.06
14	0.169	4000	0.510	4	0.335	2500	-1500	2500	2500	2500	0	4208.06
15	0.005	4000	0.632	4	0.634	2500	-1500	2500	2500	2500	0	4208.06
16	0.695	6000	0.867	5	0.660	2500	-3500	2500	2500	2500	2500	3637.45
17	0.632	6000	0.450	4	0.883	3000	-3000	3000	3000	3000	0	3849.60
18	0.551	6000	0.352	4	0.105	2000	-4000	2000	2000	2000	0	566.45
19	0.542	6000	0.604	4	0.222	2000	-4000	2000	2000	2000	0	566.45
20	0.384	4000	0.165	3	0.016	1500	-2500	1500	1500	0	0	-61.44
21	0.865	6000	0.254	3	0.109	2000	-4000	2000	2000	0	0	-740.50
22	0.400	6000	0.474	4	0.234	2000	-4000	2000	2000	2000	0	566.45
23	0.207	4000	0.886	5	0.100	2000	-2000	2000	2000	2000	2000	3709.96
24	0.635	6000	0.564	4	0.136	2000	-4000	2000	2000	2000	0	566.45
25	0.089	4000	0.341	4	0.923	3000	-1000	3000	3000	3000	0	5849.60

(7) Statistical analysis for the model outputs.

According to the simulation, the model outcomes are sorted and summarized as follows:

Outcomes	counts	probability.
less than 0	2	.08
0 - 500	0	.00
500^+ - 1000	5	.20
1000^+ - 1500	1	.04
1500^+ - 2000	1	.04
2000^+ - 2500	1	.04
2500^+ - 3000	1	.04
3000^+ - 3500	0	.00
3500^+ - 4000	9	.36
4000^+ - 4500	3	.12
4500^+ - 5000	0	.00
5500^+ - 5500	0	.00
5500^+ - 6000	2	.08
	25	1.00

Let us plot the histogram of the simulated outcomes as shown below:

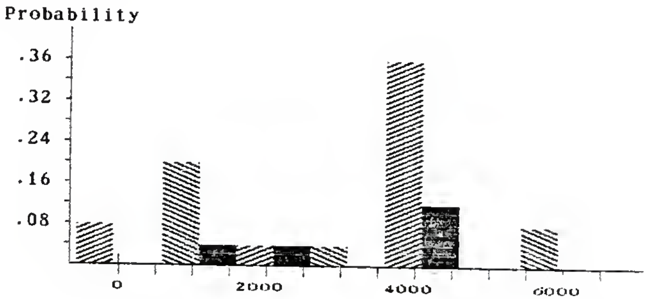


Figure 3.7 Histogram of simulation results

Of course, when numerous outcomes are simulated, the graph will be more representative of the real situation. The detail of statistical analysis should not be covered here. But we can make somewhat crude judgement that this project yields a great possibility of positive net present value, and the probable value is around $3,500 to $4,000. So, we shall accept the project.

3.3 Project Evaluation & Ranking --- under Uncertainty

Townsend [40] defined the true uncertainty as any situation in which the outcome of an event in not known with certainty. We shall assume for this moment that the uncertainty is normally **referred to as situations in which we do know :**

(1) the (finite) number of alternative possible outcomes and

(2) the 'value' of each outcome

but we do not know

(3) the probability **for the occurrence of each outcome.**

3.3.1 Laplace Principle

The philosophy of the Laplace principle, named after an early nineteenth century mathematician, is simply that if one cannot assign probabilities to the outcomes, then the outcomes should be considered equally probable.

Example 3.27

Three alternatives are considered. The possible outcomes for each alternative are shown on Table 3.13 . Under the condition of uncertainty, which alternative should be chosen by applying Laplace principle.

Table 3.13 Possible outcomes table

Cost of capital	10%	15%	20%	25%
NPV (Alt. A)	18,170	16,070	12,060	9,650
NPV (Alt. B)	19,200	14,210	11,750	8,160
NPV (Alt. C)	21,450	15,120	10,620	6,350

Solution: The equal probability assigned to each of the four possible states is 0.25.

Therefore, $E(A) = (.25)(18170 + 16070 + 12060 + 9650)$

$=13,987$

$$E(B) = (.25)(19200 + 14120 + 11750 + 8160)$$
$$= 13,330$$

$$E(C) = (.25)(21450 + 15120 + 10620 + 6350)$$
$$= 13,385$$

We shall select alternative A.

3.3.2 Maximin and Minimax Principles

The Maximin and Minimax principles hold considerable appeal for conservative or pessimistic decision makers. In applying the principle, if we are evaluating the benefit alternatives, the minimum gains associated with each alternative are determined, and the maximum value in the set of minimum values designates the alternative to be chosen. More formally stated:

(1) the Maximin principle, used in evaluating benefit alternatives, is to select the alternative associated with

$\max_j \min_k (\theta_{jk})$ where j denote the alternative
k denote the outcome
θ_{jk} the value of outcome k of alternative j

(2) the Minimax principle, used in evaluating cost alternatives, is to select the alternative associated with

$$\min_j \max_k (\theta_{jk})$$

Example 3.28

Referring to Ex. 3.27, what should be the order of ranking if

we apply the Maximin principle to the problem.

Solution: We may summarize the results by the following table

Alternative	Min NPV	Max of them
A	9,650	9,650
B	8,160	
C	6,350	

We shall select alternative A.

3.3.3 Maximax and Minimin principles

Contrary to the Maximin and Minimax principles, where decision making is extremely pesimistic, the Maximax (or Minimin) principle offers an optimistic rule for choice among projects which involves benefit (or cost). That is, the decision maker desires to select the alternative that affords the opportunity to obtain the targest value given among all probable outcomes. Formally stated,

(1) The **Maximax** principle, used in evaluating benefit alternatives, is to select the alternative associated with

$$\max_{j} \max_{k} (\theta_{jk})$$

where j denote the alternative
k denote the outcome
θ_{jk} the value of outcome k of alternative j

(2) the Minimin principle, used in evaluating cost alternatives, is to select the alternative associated with

$$\min_{j} \min_{k} (\theta_{jk})$$

Example 3.29

Referring to Ex. 3.27, when applying Maximax principle to the problem, it gives:

Alternative	Max NPV	Max of them
A	18,170	
B	19,200	
C	21,400	21,400

We shall select alternative C.

3.3.4 Hurwicz principle

The Hurwicz principle considers the decision makers' view may be neither extreme **optimism** nor extreme **pessimism**. It incorporates various levels of optimism-pessimism into the decision. The Hurwicz principle defines an index of optimism, α, scaled from 0 to 1. A value of $\alpha = 0$ indicates extreme pessimism and $\alpha = 1$ indicates extreme optimism.

Assuming that a decision maker is able to reflect his degree of optimism by assigning the index of optimism, α, then the Hurwicz value, associated with the alternative j, is defined by:

$$H_j = \alpha \left[\operatorname*{Max}_k (\theta_{jk}) \right] + (1 - \alpha) \left[\operatorname*{Min}_k (\theta_{jk}) \right]$$

for evaluating benefit alternatives

and

$$H_j = \alpha \, [\, \mathbf{Min}_k \, (\theta_{jk}) \,] + (1 - \alpha) \, [\, \mathbf{Max}_k \, (\theta_{jk}) \,] \qquad (3.26)$$

for evaluating cost alternatives

The decision will then be made to select the alternative that maximizes the Hurwicz values (for benefit alternatives) or minimizes the Hurwicz values (for cost alternatives).

Example 3.30

Referring to Ex. 3.27, suppose the decision maker set the index of optimism at $\alpha = 0.4$ level, then we can calculate the Hurwicz values, H_j's for each alternative as follows:

$$H_A = .4(18710) + (1 - .4)(9650) = 13,274$$

$$H_B = .4(19200) + (1 - .4)(8160) = 12,576$$

$$H_C = .4(21450) + (1 - .4)(6350) = 12,390$$

We shall select alternative A.

For Ex. 3.30, we can also interpret the relationship between H_j's and α by graph representation. Where,

$$H_A = \alpha(18710) + (1 - \alpha)(9650) = 9650 + 9060\alpha \; ,$$

$$H_B = \alpha(19200) + (1 - \alpha)(8160) = 8160 + 11040\alpha \; ,$$

$$H_C = \alpha(18710) + (1 - \alpha)(9650) = 6350 + 15100\alpha$$

The graph intepretation (see Figure 3.8) gives the overall decision rules to be followed with various level of α.

There are some shortcomings of the Hurwicz principle:

(1) It ignores the intermediate values of each alternative.

(2) It is unable to select a particular alternative when more than one alternative has the same Hurwicz value.

(3) The index value, α, is difficult to decide.

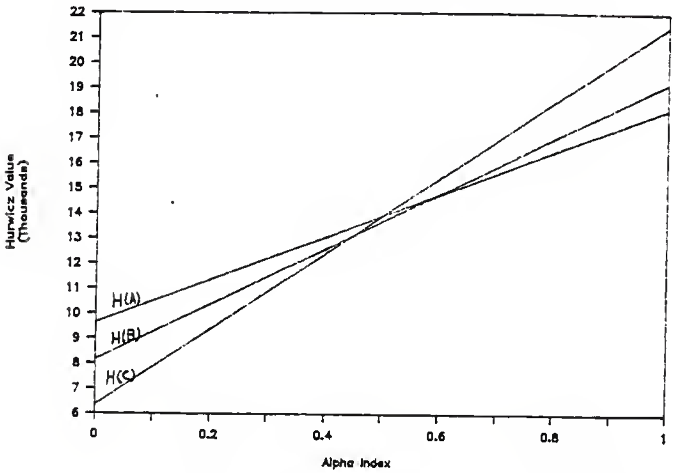


Figure 3.8 Hurwicz value versus Index of optimism (alpha)

3.3.5 Savage Principle (Minimax Regret)

The principle, proposed by L. J. Savage, introduces a quantity termed *'regret'*.The procedure for determining the regret matrix

is as follows:

(1) Denote V_{ij} as the value of alternative i at state j ; For a given state $j = k$, search for the largest gain (for benefit alternatives) or the smallest loss (for cost alternatives), which can be expressed by $\underset{j=k}{\textbf{Max}}\ V_{ij}$ (or $\underset{j=k}{\textbf{Min}}\ V_{ij}$), for all i. Let us denote the searched value as S_k .

(2) The *regret* value, R_{ik} of alternative i at a given state k is defined by

$$R_{ik} = | V_{ik} - S_k |$$

(3) Repeat steps (1) and (2) for all the other states. All the regret values can then be incorporated into a regret matrix.

In the regret matrix, the maximum regret value for each alternative is determined. The Savage principle then choose the alternative which has the minimum regret value in the set of maximum regret values.

Example 3.31

Referring to Ex. 3.27, we can construct the matrix as follows:

Table 3.14 Regret Matrix

Alternative	10%	15%	20%	25%	Max Regret
A	3280	0	0	0	3280
B	2250	1860	310	1490	2250
C	0	950	1440	3300	3300

Applying the Minimax regret principle to the problem, we shall select alternative B.

3.4 Portfolio Selection

In the security market, a portfolio indicates a group of stocks. The portfolio theory asserts that by including a large number of securities with diverse characteristics, the primary objective of an investor is to construct an efficient portfolio which can maximize the expected return of the investment in the investor's class of risk. An active portfolio needs to consider not only a single security but the whole security market. In consideration of an efficient portfolio construction, three main criteria should be measured: the expected return of the security, the variability (risk) of the security and the risk preference of the decision maker.

3.4.1 Markowitz Model

The first model to deal explicitly with risk in a portfolio sense was devised by Harry Markowitz [28] in 1952. The basis of Markowitz's approach was the use of variability of investment returns as an approximation of the risk of the investment. Markowitz used the statistical concept of variance to describe the variability of return on investments.

3.4.1.i Measurement of Risk and Return

For a single security, suppose R_s is a random variable which represents the rates of return of a certain security. Then, the expected return on the security $E(R_s)$ is

$$E(R_s) = \sum_{i=1}^{n} p_i R_i \qquad (3.27)$$

where $E(R_s)$: the expected return on the security

p_i : probability of the i_{th} return rate

R_i : the i_{th} return rate

n : number of possible returns

The risk of a single security is measured by its variance

$$Var(R_s) = \sigma_s^2 = \sum_{i=1}^{n} p_i [R_i - E(R_s)]^2 \qquad (3.28)$$

where σ_s is the standard deviation

Suppose an investor decides to invest in a portfolio which is composed of m securities. If w_i denotes the proportion (weight) of the fund invested in the i_{th} security, then the expected return of the portfolio can be expressed by:

$$E(R_p) = \sum_{i=1}^{m} w_i E(R_i) \qquad (3.29)$$

where $E(R_p)$: the expected return of the portfolio

w_i : the proportion of investment on security i

$E(R_i)$: the expected return of security i

m : the number of securities in the portfolio

The expected risk of the portfolio can be expressed by

$$E(\sigma_p^2) = \sum_{i=1}^{m} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} w_i w_j \sigma_{ij} \qquad (3.30)$$

where σ_{ij} denotes the covariance of security i and j

Let us define the correlation coefficient, ρ_{ij}, as follows:

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

then, Eq. (3.30) can be rewitten as follows:

$$E(\sigma_p^2) = \sum_{i=1}^{m} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} w_i w_j \rho_{ij} \sigma_i \sigma_j \qquad (3.31)$$

For example , the sample correlation coefficient can be expressed by

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \frac{\sum_{\text{all } i} (x_i - \bar{X})(y_i - \bar{Y})}{\sqrt{\sum_{\text{all } i} (x_i - \bar{X})} \sqrt{\sum_{\text{all } i} (y_i - \bar{Y})}} \qquad (3.32)$$

The most important properties of the sample correlation coefficient are [Devore, 08]:

(1) ρ is independent of units

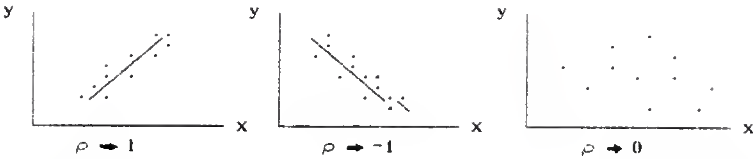
(2) $-1 \leq \rho \leq 1$

(3) $\rho = 1$ if and only if all (x_i, y_i) pairs lie on a straight line with a positive slope

(4) $\rho = -1$ if and only if all (x_i, y_i) pairs lie on a straight line with a negative slope

(5) $\rho = 0$ means no apparent relationship between **X** and **Y**

Let us explain ρ graphically as follows:



To express an individual security in relation to the market return, a single-index market model can be defined by

$$R_s = \alpha_s + \beta_s R_m + \varepsilon_s \qquad (3.33)$$

where R_s : the return of the security

α_s, β_s : constants

R_m : the market return

ε_s : error term which is assumed normally distributed with zero mean

The variance of the model can be expressed by

$$Var(R_s) = Var(\alpha_s) + Var(\beta_s R_m) + Var(\varepsilon_s)$$

Since α_s and β_s are constants, $Var(\alpha_s)$ equals to zero, and

$$Var(R_s) = \beta_s^2 \, Var(R_m) + Var(\varepsilon_s)$$

or

$$\sigma_s^2 = \beta_s^2 \sigma_m^2 + \sigma_\varepsilon^2 \qquad (3.34)$$

Eq. (3.34) can be interpreted as

Total Risk = System Risk + Unsystem Risk

σ_s^2 $\qquad$ $\beta_s^2 \sigma_m^2$ $\qquad$ σ_ε^2

The system risk ($\beta_s^2 \sigma_m^2$) is inherent in the market and therefore cannot be diversified away. The unsystem risk (σ_ε^2), can be eliminated by selecting a large number of securities

with diverse characteristics. The procedure is known as *diversification* through the construction of an efficient portfolio.

Assuming that equal amounts are invested in N securities, the proportion invested in each security is $\frac{1}{N}$. The unsystem risk of the investment can be given by

$$\sigma_{\varepsilon}^{2} = \sum_{i=1}^{N} \frac{1}{N^2} \sigma_{\varepsilon_i}^{2} = \frac{1}{N} \sum_{i=1}^{N} \frac{\sigma_{\varepsilon_i}^{2}}{N} = \frac{1}{N} \bar{\sigma}_{\varepsilon}^{2} \qquad (3.35)$$

where $\bar{\sigma}_{\varepsilon}^{2}$ is the average system risk.

We can see from Eq. (3.35) that $\sigma_{\varepsilon}^{2} \to 0$, as $N \to \infty$. So, it proves that diversification can dilute the unsystem risk. In Markowitz model, we always assume that the unsystem risk is eliminated by diversification.

Example 3.32

Two securities A and B have the expected returns and their associated subjective probabilities as follows:

Security A		Security B	
Prob.	E(R)	Prob.	E(R)
.05	30%	.15	25%
.70	15	.45	15
.15	0	.20	-5
.10	-20	.10	-10

Assume $\rho_{AB} = .7$

Assume that the investor decides to put 60% weight on stock A and 40% on stock B. Determine (1) the individual risk and return of the securities, (2) the portfolio risk and return.

solution:

(1) For individual security

$E(R_A) = .05(30) + .7(15) + .1(-20) = 10\%$

$E(R_B) = .15(25) + .45(15) + .2(-5) + .1(-10) = 8.5\%$

$\sigma_A^2 = .05(.3 - .1)^2 + \ldots + .1(-.2 - .1)^2 = .01425$

$\sigma_B^2 = .15(.3 - .085)^2 + \ldots + .1(-.1 - .085)^2 = .01305$

and $\sigma_A = .1194$ $\sigma_B = .1142$

(2) For the portfolio

$E(R_p) = .6(10) + .4(8.5) = 9.4\%$

$$E(\sigma_p^2) = (.6)^2(.01425) + (.4)^2(.01305)$$
$$+ 2(.6)(.4)(.7)(.1194)(.1142)$$
$$= .0118 \qquad \text{and} \quad \sigma_p = .1086$$

The basic assumptions underlying the Markowitz model can be summarized as follows [30]:

(1) The two relevant characteristics of a portfolio are its expected return and its **riskiness**.

(2) Rational investors will choose to hold efficient portfolio which maximize expected return for a given degree of risk.

(3) It is theoretically possible to identify efficient portfolios by analyzing each security according to its expected return, the variance of that return, and the relationship between its return and the returns of every other security in the portfolio.

3.4.1.2 Efficient Frontier

Successful application of Markowitz model depends on the ability of an investor to identify a set of efficient portfolios with a special risk associated with each portfolio. The set of efficient portfolio has come to be known as the *efficient frontier*.

Recalling Eq. (3.29) and (3.30), for the case of two securities A and B, the portfolio variance can be expressed as

$$\sigma_P = (w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_{AB})^{1/2}$$

or, $$\sigma_P = \left[w_A^2 \sigma_A^2 + (1 - w_A)^2 \sigma_B^2 + 2w_A(1 - w_A)\sigma_{AB} \right]^{\frac{1}{2}} \qquad (3.35)$$

The portfolio expected return can be expressed by

$$\bar{R}_P = w_A \bar{R}_A + w_B \bar{R}_B$$

or $$\bar{R}_P = w_A \bar{R}_A + (1 - w_A) \bar{R}_B \qquad (3.36)$$

In order to construct the relationship between $\bar{R}_P$ and σ_P, let us consider the following cases:

<u>Case 1: perfect positive correlation ($\rho = +1$)</u>

$$\sigma_P^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_A \sigma_B = (w_A \sigma_A + w_B \sigma_B)^2$$

Therefore,

$$\sigma_P = w_A\sigma_A + w_B\sigma_B = w_A\sigma_A + (1 - w_A)\sigma_B$$

we can get that

$$w_A = \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}$$

By substituting w_A into Eq. (3.36), we may obtain

$$\bar{R}_P = \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\bar{R}_A + \left[1 - \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\right]\bar{R}_B$$

$$= \left[\bar{R}_B - \frac{\bar{R}_A - \bar{R}_B}{\sigma_A - \sigma_B}\sigma_B\right] + \left[\frac{\bar{R}_A - \bar{R}_B}{\sigma_A - \sigma_B}\right]\sigma_P \quad (3.37)$$

Eq. (3.37) shows the linear relationship between expected portfolio return and the standard deviation of the portfolio. For example, a portfolio with 2 securities has the following characteristics:

	E(R)	σ
security A	14%	6%
security B	8%	3%

According to Eq. (3.37), the relationship between $\bar{R}_p$ and σ_p yields $\qquad R_p = .02 + 2\sigma_p$

Case 2: Perfect negative correlation ($\rho = -1$)

similar to case 1, we can get that

$$\sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 - 2w_A w_B \sigma_A \sigma_B = (w_A\sigma_A - w_B\sigma_B)^2$$

Therefore,

$$\sigma_P = w_A\sigma_A - w_B\sigma_B = w_A\sigma_A - (1 - w_A)\sigma_B$$

or

$$\sigma_P = -w_A\sigma_A + w_B\sigma_B = -w_A\sigma_A + (1 - w_A)\sigma_B$$

Substituting the above equation into Eq. (3.36) yields

$$\bar{R}_P = \left[\bar{R}_B + \frac{\bar{R}_A - \bar{R}_B}{\sigma_A + \sigma_B}\sigma_B\right] + \left[\frac{\bar{R}_A - \bar{R}_B}{\sigma_A + \sigma_B}\right]\sigma_P \quad (3.38)$$

or

$$\bar{R}_P = \left[\bar{R}_B + \frac{\bar{R}_A - \bar{R}_B}{\sigma_A + \sigma_B} \sigma_B \right] - \left[\frac{\bar{R}_A - \bar{R}_B}{\sigma_A + \sigma_B} \right] \sigma_P \qquad (3.39)$$

For the same example in case 1, when $\rho = -1$, the relationship between $\bar{R}_P$ and σ_P can be expressed by

$$\bar{R}_P = .1 \pm \frac{2}{3} \sigma_P$$

<u>Case 3: zero correlation ($\rho = 0$)</u>

$$\sigma_P^2 = w_A^2 \sigma_A^2 + w^2 \sigma^2 = w^2 \sigma^2 + (1 - w^2) \sigma^2$$

The relationship between $\bar{R}_P$ and σ_P is no longer linear. To consider the graphical representation of the risk and return in general cases, there is a point in the figure that worth special attention: the point that has the minimum risk. Recall Eq. (3.30) that

$$\sigma_P = \left[w_A^2 \sigma_A^2 + (1 - w_A)^2 \sigma_B^2 + 2 w_A (1 - w_A) \rho_{AB} \sigma_A \sigma_B \right]^{\frac{1}{2}}$$

$$\frac{\partial \sigma_P}{\partial w_A} = \frac{2 w_A \sigma_A^2 - 2\sigma_B^2 + 2 w_A \sigma_B^2 + 2 \sigma_A \sigma_B \rho_{AB} - 4 w_A \sigma_A \sigma_B \rho_{AB}}{\left[w_A^2 \sigma_A^2 + (1 - w_A)^2 \sigma_B^2 + 2 w_A (1 - w_A) \sigma_{AB} \right]^{\frac{1}{2}}}$$

Setting the equation to zero, we may get

$$w_A = \frac{\sigma_B^2 - \sigma_A \sigma_B \rho_{AB}}{\sigma_A^2 + \sigma_B^2 - 2 \sigma_A \sigma_B \rho_{AB}} \qquad (3.40)$$

In the case of $\rho_{AB} = 0$, the minimum risk occurs at

$$w_A = \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}$$

For the same example as case 1,

$$\sigma_P = [\ (.6)^2\ w_A^2 + (.3)^2(1 - w_A)^2\ ]^{\frac{1}{2}}$$

$$= [\ .0045\ w_A^2 - .0018\ w_A + .09\ ]^{\frac{1}{2}}$$

The minimum risk occurs at

$$w_A = \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2} = \frac{.3^2}{.6^2 + .3^2} = 0.2$$

<u>Case 4 : Intermediate correlation ($\rho = 0.5$)</u>

The correlation between any two stocks is always greater than 0 and less than 1. To show a more typical relationship between risk and return for two stocks, we have chosen to examine the relationship when $\rho_{AB} = 0.5$.

For the example of case 1 of stock A and B

$$\sigma_P = \left[(.06)^2 w_A^2 + (.03)^2 w_B^2 + 2w_A(1 - w_A)(.5)(.06)(.03)\right]^{\frac{1}{2}}$$

$$= (\ .0027w_A^2 + .0009\ )^{\frac{1}{2}}$$

The minimum risk occurs at

$$w_A = \frac{(.03)^2 - (.5)(.06)(.03)}{(.03)^2 + (.06)^2 - 2(.5)(.06)(.03)} = 0$$

Table 3.15 shows the results of $\bar{R}_P$ and σ_P with different weights and various correlation levels. The results in Table 3.15 can be represented graphically by Figure 3.9.

Figure 3.9 shows a typical relationship between portfolio expected return and risk. It is a reasonable assumption that

the correlation coefficients between stocks are normally positive values. Therefore, in Figure 3.9, only the region between $\rho = 0$ and $\rho = 1$ is considered feasible. In general, the efficient portfolios will always construct a graph as shown in Figure 3.10.

Table 3.15 Expected Value & Standard Deviation table

$E(R_A) = 14\%$ $E(R_B) = 8\%$ $\sigma_A = 6\%$ $\sigma_B = 3\%$

$$E(R_P) = w_A E(R_A) + (1 - w_B) E(R_B)$$

weight	E(R)	Std. Deviation of portfolio return				
		$\rho_{AB} = 1$	-1(a)	-1(b)	0	.5
0.0	0.080	0.030	-0.030*	0.030	0.030	0.030
0.1	0.086	0.033	-0.021*	0.021	0.028	0.030
0.2	0.092	0.036	-0.012*	0.012	0.027	0.032
0.3	0.098	0.039	-0.003*	0.003	0.028	0.034
0.4	0.104	0.042	0.006	-0.006*	0.030	0.036
0.5	0.110	0.045	0.015	-0.015*	0.034	0.040
0.6	0.116	0.048	0.024	-0.024*	0.038	0.043
0.7	0.122	0.051	0.033	-0.033*	0.043	0.047
0.8	0.128	0.054	0.042	-0.042*	0.048	0.051
0.9	0.134	0.057	0.051	-0.051*	0.054	0.056
1.0	0.140	0.060	0.060	-0.060*	0.060	0.060

* negative σ_P is not feasible since we assume positive risk

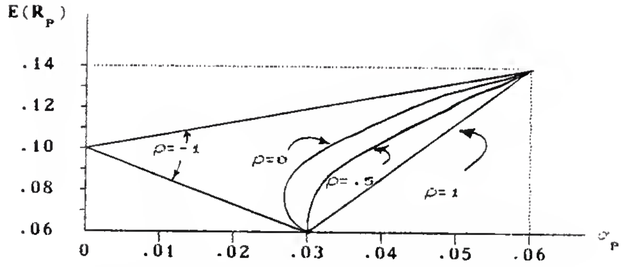


Figure 3.9 Efficient Frontier

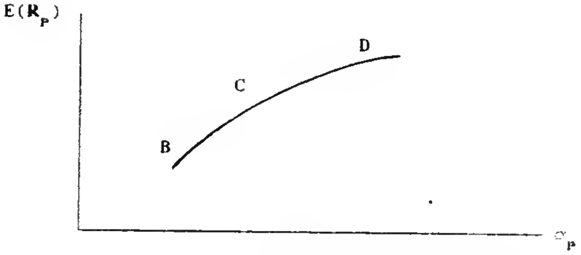


Figure 3.10 Efficient Frontier

We can see that all the other portfolios are dominated by the portfolios lying on the curve BCD. Such curve, BCD, which is convex to the vertical axis, is known as the *efficient frontier*.

3.4.2 Lower-Confidence-Limit (LCL) Criterion

When the number of securities is large, the efficient

portfolio set might involve a substantial number of combinations to choose. The use of lower-confidence-limit model can reduce the problem to a manageable size. The lower-confidence-limit (L) is defined by

$$L = E(R_p) - k\,\sigma_p \qquad (3.41)$$

The number k is a constant chosen by the investor and refers to the number of standard deviations in a normally distributed portfolio return. The concept can be shown graphically in Figure 3.11.

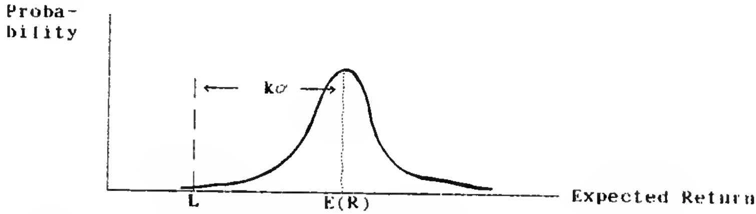


Figure 3.11 Lower-Confidence-Limit in a Normal Distribution

L is the minimum acceptable level for portfolio return. By assuming normal distribution for portfolio return, when $k = 2$, it means that decision maker is willing to accept only the 2.5% probability of return falling below L.

<u>Example 3.33</u>

An investor is evaluating the following 4 efficient portfolios which are characterized as follows:

Portfolio	E(R)	σ_P
A	26%	8%
B	30	11
C	15	4
D	40	14

Suppose the decision maker is willing to accept .025 chance of the return falling a certain L, (ie k = 2); determine the portfolio **to be selected.**

Solution:

$L_A = .26 - (2)(.08) = .10$

$L_B = .30 - (2)(.11) = .08$

$L_C = .15 - (2)(.04) = .08$

$L_D = .40 - (2)(.14) = .12$

Therefore, portfolio D should be selected.

3.4.3 Capital Asset Pricing Model (CAPM)

The major difficulty in Markowitz model is that it requires tremendous calculations to obtain the statistical figures. In 1963, William Sharp [36] suggested that because all stocks are correlated with the market, the relationship of each security to the market could be act as a surrogate for the covariances for each security relative to other securities. The model is termed Capital Asset Pricing Model (CAPM), which represents the price of immediate consumption and the price of risk.

One simplification of Markowitz Model, the single-index market

model, can be expressed as follows:

$$R_S = \alpha_S + \beta_S R_M + \varepsilon_S \qquad (3.42)$$

where R_S : the total return of the stock

α : constant, the risk-free return

$\beta_S R_M$: represents the systematic return

As we have discussed in section 3.4.1, the system risk associated with the market model is $\sigma_S^2 = \beta_S^2 \sigma_M^2$, therefore, $\sigma_S = \beta_S \sigma_M$, where σ_M is the standard deviation of the portfolio market. While in equilibrium, σ_M can be assumed a constant. Therefore, β_S can be considered a risk measure of the portfolio. Solving the market model by linear regression, we may get that

$$\beta_S = \frac{Cov(R_S, R_M)}{\sigma^2_{R_M}} \qquad (3.43)$$

For the market security, β_M is

$$\beta_M = \frac{Cov(R_M, R_M)}{\sigma^2_{R_M}} = \frac{\sigma^2_{R_M}}{\sigma^2_{R_M}} = 1$$

and for the risk-free asset, β_f is

$$\beta_f = \frac{Cov(R_f, R_M)}{\sigma^2_{R_M}} = \frac{0}{\sigma^2_{R_M}} = 0$$

If an investor invests a proportion (w) of his fund in the risky portfolio, the market portfolio, and the other portion

(1-w) in the riskless assets. Then, the new portfolio risk, β_p, can be measured by

$$\beta_p = w\,\beta_M + (1 - w)\,\beta_f = w\,(1) + (1 - w)\,(0) = w$$

Therefore, the risk can be measured by the fraction of the fund invested in the risky portfolio. Also, let us recall that

$$\bar{R}_p = w\,\bar{R}_M + (1 - w)\,R_f \qquad (3.44,\ \text{recall } 3.29)$$

By replacing w with β_p, we may obtain

$$\bar{R}_p = \beta_p \bar{R}_M + (1 - \beta_p)\,R_f = R_f - \beta_p R_f + \beta_p \bar{R}_M$$

Also,

$$\bar{R}_p = R_f + \beta_p\,(\bar{R}_M - R_f) \qquad (3.45)$$

Eq. (3.45) represents the CAPM. It says that in the equilibrium, investors will price capital assets so that the expected return of a portfolio is equal to risk-free return rate plus a risk premium return rate that is proportional to the risk measure β.

Example 3.34

A portfolio has $\beta = 2$, and an assumed risk free rate of 6%. The CAPM will be

$$\bar{R}_p = 6\% + 2\,(\bar{R}_M - 6\%) = -6\% + 2\,\bar{R}_M$$

When the market expected return is 5%, the portfolio expected return is

$$\bar{R}_p = -6\% + 2(5\%) = 4\%$$

The following assumptions are implicil in CAPM [Mittra, 30]

(1) All investors can borrow or lend at a given riskless rate and there are no ristrictions on short sales of any assets.

(2) All investors choose portfolios on the basis of their single period mean and variance of return.

(3) All investors have identical subjective estimates of joint probability distribution on the return of all assels.

(4) All assets are perfectly liquid and divisible.

(5) The quantities of all assets are given.

(6) All investors are risk takers.

(7) The markets are in equilibrium. That is , all securities are perfectly priced.

3.4.4 Capital Market Line (CML)

The introduction of riskless assets into portfolios considerably simplifies the portfolio analysis. Riskless lending can be considered as investment in an asset with cerlain outcomes, such as government bills or saving accounts. Riskless borrowing can be considered as selling an asset with a certain value, such as loan or selling bonds. The expected return on the combinations of riskless assets and risky portfolio is given by

$$\beta_P = w\,\beta_M + (1 - w)\,\beta_f$$

The risk on the combination is

$$\sigma_p = \left[w^2 \sigma_M^2 + (1 - w)^2 \sigma_f^2 + 2w(1 - w) \rho_{Mf} \sigma_M \sigma_f \right]^{\frac{1}{2}}$$

since $\sigma_f = 0$, therefore

$$\sigma_p = (w^2 \sigma_M^2)^{\frac{1}{2}} = w \sigma_M$$

So, $w = \dfrac{\sigma_p}{\sigma_M}$

Substituting w of Eq. (3.44), we may get

$$\bar{R}_p = \left(1 - \frac{\sigma_p}{\sigma_M}\right) R_f + \frac{\sigma_p}{\sigma_M} \bar{R}_M$$

or

$$\bar{R}_p = R_f + \left(\frac{\bar{R}_M - R_f}{\sigma_M}\right) \sigma_p \qquad (3.46)$$

Eq.(3.46) is an equation of straight line, the capital market line (CML), which can be represented by Figure 3.12.

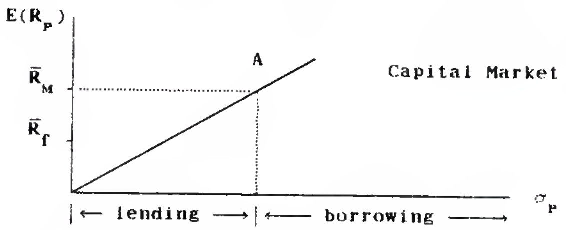


Figure 3.12 The Capital Market Line

The region to the left of A are lending portfolios since a certain portion of the investment is placed in risk-free assets. The region to the right of A are borrowing portfolios since the decision maker seek higher return with higher risk. He will borrow funds at risk-free rate and invest in

higher risk securities.

The investors who have faced the efficient frontier and riskless lending and borrowing rate, shown in Figure 3.13, would hold the portfolio G in the efficient frontier.

The point G is the tangency point between the CML and the efficient frontier. The reason to hold G is that all combinations along the CML line, R_fG, are superior to all the other combinations, such as R_fB, for they offer greater return for the same risk.

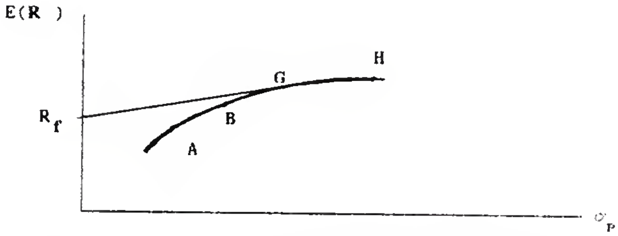


Figure 3.13 CML and the Efficient Frontier

An equivalent way to identify R_fG is to recognize that it is the line connecting the efficient frontier with the greatest slope. The slope of the CML line is

$$\theta = \frac{\bar{R}_M - R_f}{\sigma_M} \qquad (3.47)$$

While considering a portfolio with combinations of securities,

the efficient set is determined by finding the portfolio with the greatest ratio of excess return to the standard deviation of the return, and also satisfies the constraint that the proportions of all securities sum up to 1. The concept can be formulated by the model as follows:

$$\text{OBJ} \quad \text{Max} \quad \theta = \frac{\bar{R}_M - R_f}{\sigma_M}$$

$$\text{S.T.} \quad \sum_{i=1}^{n} w_i = 1$$

$$w_i \geq 0, \text{ for } i = 1,2,\ldots,n$$

where w_i is the proportion of fund placed on security i

This is a constrained nonlinear maximization problem, which can be solved by the method of Largrange multipliers. An alternative approach is to substitute the constraint into the objective function and solve the objective function as if it were an unconstrained problem by classical calculus. We know

$$\bar{R}_p = \sum_{i=1}^{n} w_i \bar{R}_i \text{ , also, } R_f = (1)R_f = (\sum_{i=1}^{n} w_i)R_f = \sum_{i=1}^{n} w_i R_i$$

Therefore,

$$\bar{R}_p - R_f = \sum_{i=1}^{n} w_i(\bar{R}_i - R_f) \qquad (3.48)$$

The complete form of θ can then be expressed as

$$\theta = \frac{\sum_{i=1}^{n} w_i(\bar{R}_i - R_f)}{\left[\sum_{i=1}^{n} w_i^2\ \sigma_i^2 + 2\sum_{i=1}^{n-1}\ \sum_{j=i+1}^{n} w_i w_j\ \sigma_{ij}\right]^{\frac{1}{2}}}$$

By using the product rule and the chain rule of differentiation, we can obtain the optimum w_i's by performing the following steps:

$$\frac{d\theta}{dw_k} = \left(\sum_{i=1}^{n} w_i(\bar{R}_i - R_f)\right)\left[\left(\frac{-1}{2}\right)\left(\sum_{i=1}^{n} w_i^2\ \sigma_i^2 + 2\sum_{i=1}^{n-1}\ \sum_{j=i+1}^{n} w_i w_j\ \sigma_{ij}\right)^{-\frac{3}{2}}\right.$$

$$\left.\left(2w_k\sigma_k^2 + 2\sum_{\substack{j=1\\ j\neq k}}^{n} w_j\sigma_{kj}\right)\right] + \left(\sum_{i=1}^{n} w_i^2\ \sigma_i^2 + 2\sum_{i=1}^{n-1}\ \sum_{j=i+1}^{n} w_i w_j\ \sigma_{ij}\right)^{\frac{1}{2}}$$

$$\left(\bar{R}_k - R_f\right) = 0$$

Let us multiply the derivate by

$$\left[\sum_{i=1}^{n} w_i^2\ \sigma_i^2 + 2\sum_{i=1}^{n-1}\ \sum_{j=i+1}^{n} w_i w_j\ \sigma_{ij}\right]^{\frac{1}{2}}$$

which yields

$$-\left[\frac{\sum_{i=1}^{n} w_i(\bar{R}_i - R_f)}{\sum_{i=1}^{n} w_i^2\ \sigma_i^2 + 2\sum_{i=1}^{n-1}\ \sum_{j=i+1}^{n} w_i w_j\ \sigma_{ij}}\right]\left(w_k\sigma_k^2 + \sum_{\substack{j=1\\ j\neq k}}^{n} w_j\sigma_{kj}\right) + \left(\bar{R}_k - R_f\right) = 0 \qquad (3.49)$$

Let us define λ by

$$\lambda = \left[\frac{\sum_{i=1}^{n} w_i(\bar{R}_i - R_f)}{\sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_i w_j \sigma_{ij}} \right]$$

then Eq. (3.49) becomes

$$-\lambda \left(w_k \sigma_k^2 + \sum_{\substack{j=1 \\ j \neq k}}^{n} w_j \sigma_{kj} \right) + \left(\bar{R}_k - R_f \right) = 0$$

or

$$-\left(\lambda w_k \sigma_k^2 + \lambda \sum_{\substack{j=1 \\ j \neq k}}^{n} w_j \sigma_{kj} \right) + \left(\bar{R}_k - R_f \right) = 0 \qquad (3.50)$$

Let $Z_k = \lambda w_k$. We can see that Z_k is proportional to w_k. Therefore, we can establish n simultaneous equations. By solving the simultaneous equations, we can obtain Z_i's, which are the optimum values of the model. Then, w_i's can be determined by using the following normalization equation:

$$w_k = \frac{Z_k}{\sum_{i=1}^{n} Z_i} \qquad (3.51)$$

Let us illustrate this application by the following example:

Example 3.35 [22]

Consider three securities which have characteristics summarized as follows:

Security	$\bar{R}$	σ
1	14%	6%
2	8	3
3	20	15

Assume that $\sigma_{12} = .5$, $\sigma_{13} = .2$ and $\sigma_{23} = .4$ and the risk-free **lending and borrowing rate is 5%. How are we to construct an efficient strategy for this portfolio ?**

Solution: According to Eq. (3.50), we can establish the following simultaneous equations:

$$\bar{R}_1 - R_f = Z_1 \sigma_1^2 + Z_2 \sigma_{12} + Z_3 \sigma_{13}$$
$$\bar{R}_2 - R_f = Z_2 \sigma_2^2 + Z_1 \sigma_{21} + Z_3 \sigma_{23}$$
$$\bar{R}_3 - R_f = Z_3 \sigma_3^2 + Z_1 \sigma_{13} + Z_2 \sigma_{32}$$

or

$$14 - 5 = 36\, Z_1 + (.5)(6)(3)\, Z_2 + (.2)(6)(15)\, Z_3$$
$$8 - 14 = (.5)(6)(3)\, Z_1 + 9\, Z_2 + (.4)(3)(15)\, Z_3$$
$$20 - 5 = (.2)(6)(15)\, Z_1 + (.4)(3)(15)\, Z_2 + 225\, Z_3$$

Simplifying the above equations yields

$$4\, Z_1 + Z_2 + 2\, Z_3 = 1$$
$$3\, Z_1 + 3\, Z_2 + 6\, Z_3 = 1$$
$$6\, Z_1 + 6\, Z_2 + 75\, Z_3 = 1$$

The solutions of Z_i's are

$$Z_1 = \frac{14}{63},\quad Z_2 = \frac{1}{63},\quad Z_3 = \frac{3}{63} \quad \text{and} \quad \Sigma Z_i = \frac{18}{63} = T$$

Therefore, the optimum proportions of fund allocated on each security are

$$w_1 = \frac{Z_1}{T} = \frac{14}{18},\quad w_2 = \frac{1}{18} \quad \text{and} \quad w_3 = \frac{3}{18}$$

3.4.5 Security Market Line (SML)

The capital market line (CML) relates total expected portfolio risk to the expected portfolio return. The beta value relates indivudual security risk to the market risk. Also, the beta measures the systematic risk of a security. In addition to CML, the security market line (SML) uses beta as independent variable related to the expected return of a portfolio. The difference between CML and SML should be noted. The CML is a linear relationship between expected return of the portfolio and the total risk (systematic and nonsystematic) associated with it. The SML is the linear relationship between expected return and beta (the sustematic risk) on which both portfolio and individual security can lie. SML can be expressed as follows:

$$\bar{R}_P = R_f + \beta_P (\bar{R}_M - R_f) \qquad (3.45, \text{ recalled })$$

Figure 3.14 represents a SML which always has a positive slope, indicating that the expected return increase with risk.

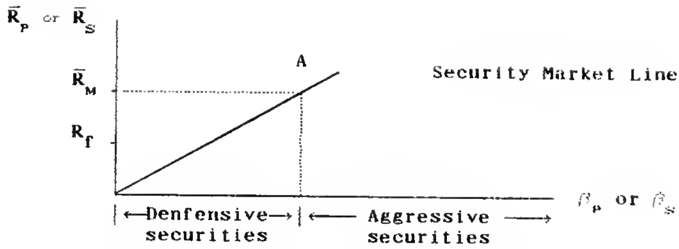


Figure 3.14 The Security Market Line

As we have mentioned , β equals 1 for the market portfolio.

Example 3.36

Suppose the short-term risk-free interest rate is 9% and the expected return on the market, $\bar{R}_M$, is 14%; then SML can be expressed as follows:

$$\bar{R}_s = 9\% + (14\% - 9\%)\ \beta_s \quad = \quad 9\% + 5\%\ \beta_s$$

With the model above, we can determine the expected portfolio return according to decision makers' risk preference. If the beta is set at .5 level, then the expected return is

$$\bar{R}_s = 9\% + 5\%\ (.5) = 11.5\%$$

The purpose of the SML construction is to offer the decision makers a cornerstone for developing effective portfolio strategies. Normally, when a security lies above the SML, it is undervalued, so it should be considered a candidate to buy. Similarily, when a security lies below SML, it is overvalued

and should be considered a candidate for sale.

With the SML concept, Mittra presents the active investment strategies to construct an effective portfolio by applying technical and analytical skills. The strategies are described by the following steps:

Step 1. Construct the stock universe

A stock universe is the group of securities that decision makers are to analyze. This is the construction of an initial portfolio.

Step 2. The expected return

The expected return associated with each stock can be determined by the security analysts through calculations. For example, the price of a stock cna be calculated by

$$PV = \sum_{i=1}^{n} \frac{D_n}{(1 + r)^n} + \frac{P_N}{(1 + r)^N}$$

where PV : the present value

D_n : dividend at the end of period n

P_N : holding price

r : the hurdle rate

N : holding period

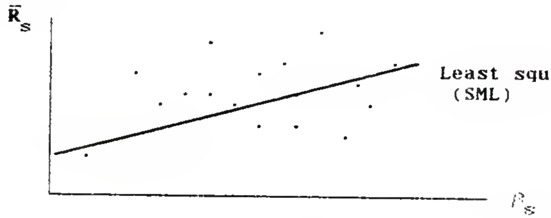
Step 3. Stock beta

Many service companies publish betas of most of the stocks in the market. For example, Wilshire Associations, Inc.,

Rosenberg Beta Services and Merrill Lynch. The security analysts may obtain betas of all stocks from the published sources.

Step 4A. SML — basic approach

We can plot on the graph the expected return associated with beta points for all securities of the stock universe. Then, the least squared line can be determined, which is approximate the SML line.



Step 4B. SML — alternative approach

The SML line developed by step 4A may have upward or downward bias if some of the securities have extraordinary high or low expected returns in the stock universe. To avoid the bias, one alternative is to divide the stock universe into n equal partitions, with each partition containing approximately the same number of stocks. Then, the median expected returns and median betas of each partition can be calculated. Finally, a modified SML can be obtained on the basis of the median

points. Figure 3.15 shows the partitions when n = 5, the least squared line is determined by the 5 median points.

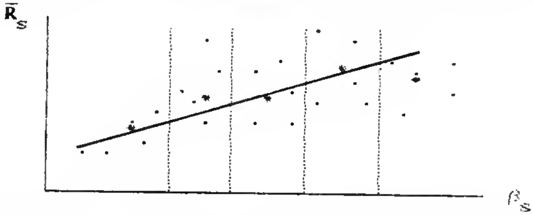


Figure 3.15 SML — alternative approach

Step 5. The decision band

Because of the fact that the expected returns of stocks vary purely due to the short-term market fluctuations, it is proper to replace the SML with decision range instead of merely the SML line. The range upward and downward the SML with a certain confidence interval can be obtained analytically by statistical calculations [Devore, 08, pp. 441-6] for linear regression model. The statistical approach is tedious and time-consuming. Therefore, subjective decision on the range is usually imposed according to decision makers' experience. The **limit of the decision range, called the 'Buy and Sell' line, may be specified; for example, 2% above and below the SML.**

Let us see Figure 3.16, which represents the decision band as the reference for portfolio strategies. Securities located

above the 'Buy' Line should be purchased since they offer higher return than expected. Securities lying below the 'Sell' line are recommended for sale since they are overvalued.

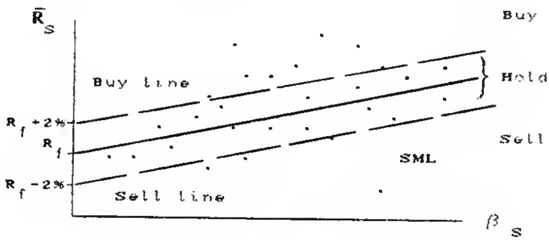


Figure 3.16 Decision band representation

Step 6. Investor's risk preference

The decision maker must determine his risk preference, the beta value, toward the security market.

Step 7. Portfolio construction

To optimize the portfolio selection problem, a linear programming model is incorporated to find the optimum proportions (w_i's) of fund placed on each security so that the expected return is maximized under the constraints of the investor's risk preferences. The LP model can be formulated in matrix form as follows:

$$\begin{aligned} \text{Max} \quad & \underline{W}^T\underline{R} \\ \text{S.T} \quad & A\underline{W} \le \underline{b} \\ & \underline{W} \ge \underline{0} \end{aligned}$$

where $\underline{W}^T$: the transpose of weight vector

$\underline{R}$: the expected return vector

$\underline{A}$: the $m \times n$ coefficient matrix

$\underline{b}$: the RHS vector

The results of the LP model are the optimum allocations of investor's fund on the stock universe. Let us demonstrate the investment strategies by Example 3.37.

Example 3.37

The Portfolio Manager (PM) of Jay company is considering an investment on the stock market. The following steps show his approach for the investment strategies.

Step 1. The Stock universe

The PM decides to invest three different markets: the computer & communication, the oil & petroleum and the retailing. The following stocks are considered as the stock universe in this example.

A. Computer & Communication	B. Oil & Petroleum	C. Retailing
1. AT&T	11. EXXON	16. WAL MART
2. BURROUGH	12. MARATH	17. K MART
3. DATA POINT	13. PHILLIPS	18. J C PENNEY
4. DIGITAL	14. STANDARD	19. REVCO
5. HEWLETT-PK	15. TEXACO	20. SEARS
6. IBM		
7. INTEL		
8. MORTOROLA		
9. TEXAS INS.		
10. UTD TeleCM		

Step 2. Expected Returns and Step 3. Stock betas

The security analysts calculate the expected return of the stocks in the stock universe. As mentioned previously, beta values are obtained from published sources. Table 3.16 shows the expected returns and betas of all stocks.

Table 3.16 Stock returns & betas [30, pp. 614-615]

Name	β	E(R)	Name	β	E(R)
A. Comp.& Comm.			B. Oil & Petro.		
1. AT&T	0.51	14.7%	11. EXXON	0.74	15.5%
2. BURROUGH	1.1	15.6%	12. MARATH	0.74	16.0%
3. DATA POINT	1.88	23.4%	13. PHILLIPS	0.8	16.9%
4. DIGITAL	1.35	17.0%	14. STANDARD	0.72	15.2%
5. HEWLETT-PK	1.28	14.0%	15. TEXACO	0.71	15.4%
6. IBM	0.89	16.2%			
7. INTEL	1.85	13.1%	C. Retailing		
8. MORTOROLA	1.44	13.5%	16. WAL MART	1.36	16.4%
9. TEXAS INS.	1.24	13.3%	17. K MART	1.24	14.1%
10. UTD TeleCM	0.71	13.3%	18. J C PENNEY	1.28	13.8%
			19. REVCO	1.13	18.1%
			20. SEARS	1.44	14.4%

Step 4B. SML — alternative approach

Let us assume that we are dealing with a very big market. In

order to avoid the bias due to outliers, the data set is partitioned into 5 quantiles with each quantile containing 4 stocks. Table 3.17 shows the information **for each quantile.**

Table 3.17 Stock universe with 5 quantiles

Name	sorted β	E(R)	Name	sorted β	E(R)
Quantile 1:			Quantile 4:		
1. AT&T	0.51	14.7%	5. HEWLETT-PK	1.28	14.0%
15. TEXACO	0.71	15.4%	18. J C PENNEY	1.28	13.8%
10. UTD TeleCM	0.71	13.3%	4. DIGITAL	1.35	17.0%
14. STANDARD	0.72	15.2%	16. WAL MART	1.36	16.4%
Quantile 2:			Quantile 5:		
11. EXXON	0.74	15.5%	8. MORTOROLA	1.44	13.5%
12. MARATH	0.74	16.0%	20. SEARS	1.44	14.4%
13. PHILLIPS	0.80	16.9%	7. INTEL	1.85	13.1%
6. IBM	0.89	16.2%	3. DATA POINT	1.88	23.4%
Quantile 3:					
2. BURROUGH	1.10	15.6%			
19. REVCO	1.13	18.1%			
17. K MART	1.24	14.1%			
9. TEXAS INS.	1.24	13.3%			

The medians of the β and $\bar{R}$ in each quantile can be determined by calculating the individual sorted data set of β's and $\bar{R}$'s in the quantile. Table3.18, shows the median points of each quantile.

β sorted	median β	$\bar{R}$ sorted	median $\bar{R}$	β sorted	median β	$\bar{R}$ sorted	median $\bar{R}$
Q.1				Q.4			
0.51		13.3%		1.28		13.8%	
0.71	0.71	14.7%	14.95%	1.28	1.32	14.0%	15.20%
0.71		15.2%		1.35		16.4%	
0.72		15.4%		1.36		17.0%	
Q.2				Q.5			
0.74		15.5%		1.44		13.1%	
0.74	0.77	16.0%	16.10%	1.44	1.65	13.5%	13.95%
0.80		16.2%		1.85		14.4%	
0.89		16.9%		1.88		23.4%	
Q.3							
1.10		13.3%					
1.13	1.19	14.1%	14.85%				
1.24		15.6%					
1.24		18.1%					

Table 3.18 Median points of all the quantiles

With the 5 median points, the linear regression can be performed to obtain the least squared line, which can be considered the SML. The result of the regression is

$$\hat{R} = -0.0145 + 0.1664\ \beta$$

The R squared of the estimate is 0.5432, which indicates that the linear relationship between expected return and beta is not quite strong.

Step 5. The decision band

The decision maker arbitrarily sets the decision range 2% above and 2% below the SML. Figure 3.17 shows the relationship between the decision band and the securities in the universe.

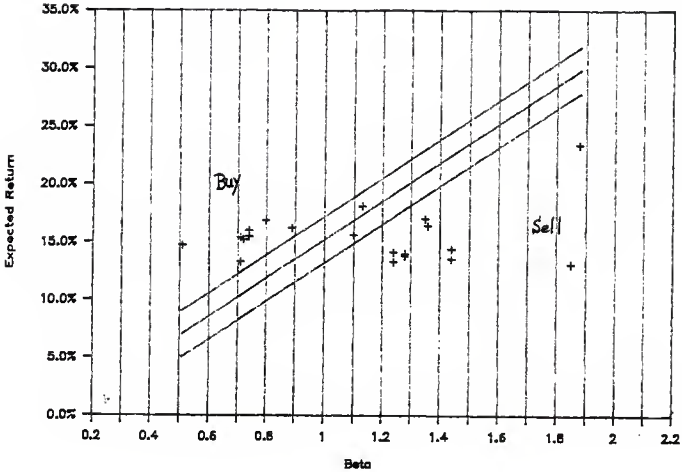


Figure 3.17 The decision band representation

All the securities above the 'Buy' line are considered candidates **for investment. The candidates are listed as:**

Name	β	E(R)
1. AT&T	0.51	14.7%
15. TEXACO	0.71	15.4%
10. UTD TeleCM	0.71	13.3%
14. STANDARD	0.72	15.2%
4. STANDARD	0.72	15.3%
11. EXXON	0.74	15.5%
12. MARATH	0.74	16.0%
13. PHILLIPS	0.80	16.9%
6. IBM	0.89	16.2%

Step 6. Investor's risk preference

Suppose the decision maker decides to take a risk , the beta value, less than 1.1. Also, he decides to diversify the investment by setting that

(1) no more than 30% of the fund can be invested in a single security,

(2) no more than 70% of the fund can be invested in the computer & communication market.

Step 7. Portfolio construction

Let us define the weights of the securities as the decision variables which are listed below:

Name	weight
1. AT&T	W_1
15. TEXACO	W_2
10. UTD TeleCM	W_3
4. STANDARD	W_4
11. EXXON	W_5
12. MARATH	W_6
13. PHILLIPS	W_7
6. IBM	W_8

The optimal portfolio construction can be obtained by solving a Linear Programming (LP) model where

$$\text{OBJ: Max} \quad 14.7W_1 + 15.4W_2 + 13.3W_3 + 15.2W_4 + 15.5W_5 + 16W + 16.9W + 16.2W$$

$$\text{S.T} \quad W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 = 1$$

$$.5W_1 + .71W_2 + .71W_3 + .72W_4 + .74W_5 + .74W_6 + .80W_7 + .89W_8 \le 1.1$$

$$W_1 + W_3 + W_8 \le 0.7$$

$$W_1 \le 0.3$$

$$W_2 \le 0.3$$

$$W_3 \le 0.3$$

$$W_4 \le 0.3$$

$$W_5 \le 0.3$$

$$W_6 \le 0.3$$

$$W_7 \le 0.3$$

$$W_8 \le 0.3$$

$$W_i \ge 0 \quad \text{for } i = 1,2,\ldots,8$$

The optimum solutions are

$$W_1 = W_2 = W_3 = W_4 = 0 ,$$

$$W_5 = 0.1, \quad W_6 = W_7 = W_8 = 0.3$$

The optimum expected return is 16.28%

3.4.6 Arbitrage Pricing Model (APM)

The CAPM is an equilibrium theory of asset pricing. One of the

most troubling problems of the model has been the CAPM's single source risk in the market. Harrington [13] declared that people may believe that the market value is not the only factor that is important in determining the return an asset is expected to earn. In addition, some other factors should be considered, such as price earnings ratio, the inflation rate, stock-issue size, liquidity, taxes and even the time of the year in which the purchases and sales of stocks occurred. Many practioners believe these factors are important and should be added to improve the CAPM model.

Arbitrage Pricing Model (APM) is a multi-factor equilibrium pricing model. The theory relates the expected return of an asset from the risk-free assets and a series of other common factors that systematically enchance or detract from the expected return. The model can be expressed as follows :

$$E(R_j) = R_f + \beta_{j1}(E(RF_1) - R_f) + \beta_{j2}(E(RF_2) - R_f) + \dots + \beta_{jn}(E(RF_n) - R_f) \qquad (3.51)$$

where R_j : the return on an asset

R_f : the risk-free rate of return

β_{jk} : the sensitivity of asset j to a factor k

RF_k : expected return of a portfolio with an average (1.0) sensitivity to a factor k

j : the asset

k : the factor

In 1980, Roll and Ross [34] derived the common factors from a set of data and then tested them from their relationships to fundamental macroeconomic variables, such as inflation, oil price and industrial production. They reported the four macro-economic variables which were important in determining the returns:

(1) unanticipated change in inflation

(2) changes in expected industrial production

(3) unanticipated changes in risk premiums

(4) unanticipated changes in the slope of the term structure of interest rates.

To apply the model to real world portfolio selection, we may employ the SML strategies (see Section 3.4.5) which use linear regression and linear programming techniques to optimize the portfolio.

3.5 Mathematical Programming (MP)

We have made assumptions in previous sections in order to simplify the project evaluation problems. The assumptions may include , for example, the project independence and the single mind to achieve the highest NPV. Real world problems are usually much more complex. The complexity **may be due to (1) project** interactions, (2) multi-attribute consideration, (3) multi-

objective consideration or (4) certain constraints.

To solve complex problems, Mathamatical Programming (MP) methods are powerful tools which can be utilized to facilitate the evaluation of alternatives. MP models are "abstracts" of real systems which try to incorporate crucial elements into the models and describe the real system quantitatively. After the models are built, certain algorithms can then be performed to solve the models.

3.5.1 Classification of MP techniques

The MP problems can be broadly classified into three categories: Single Objective Decision Making (SODM), Multiple Objective Decision Making (MODM) and Multiple **Attribute** Decision Making (MADM). Traditional operations research has offered many techniques solving SODM problems, such as linear programming (LP), nonlinear programming (NLP) and dynamic programming (DP). The **MODM techniques are for planning or** designing purposes. MODM is **characterized by** (1) a set of quantifiable objectives, (2) a set of well defined constraints and (3) a process of obtaining trade-off information [Hwang & Yoon, 16]. The **MADM techniques are for** selection or evaluation purposes. The distinctions between MADM and MODM are compared in Table 3.19.

Table 3.19 MADM vs MODM (Hwang & Yoon, [16])	MADM	MODM
criteria (defined by)	attributes	objectives
objective	implicit (ill defined)	explicit
attribute	explicit	implicit
constraints	inactive (incorporated into attributes)	active
alternative	finite, discrete	infinite, continuous
interactive with DM	not much	mostly
usage	selection	design

Let us discuss more about the three categories:

1. <u>Single Objective MP techniques</u>

Techniques in this category include:

(a) <u>Linear Programming (LP)</u>

The LP can be expressed by the following matrix form

$$\begin{aligned} \text{OBJ} \quad \text{Max/Min} \quad & \underline{C}^T \underline{X} \\ \text{S.T} \quad & A\,\underline{X} \le \underline{b} \\ & \underline{X} \ge \underline{0} \end{aligned}$$

where $\underline{C}$: the coefficient vector

$\underline{X}$: the decision variable vector

A : a $m \times n$ matrix of coefficients

$\underline{b}$: the RHS column vector

$\underline{0}$: the zero vector

Simplex method algorithm is used to solve the LP problems.

(b) NonLinear Programming (NLP)

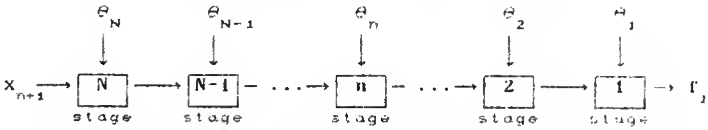
The NLP has the similar matrix form as LP, except that at least one equation in it is nonlinear. NLP problems are more complicated. When all equations in NLP are differentiable, the Kuhn-Tucker condition approach is the typical way to solve the problem. Quadratic and geometric programming are two special cases in NLP which can be solved by certain algorithms. For general NLP problem, searching techniques are used for solutions.

(c) Zero-one Programming (ZP) and Integer Programming (IP)

In this category, some or all of the decision variables must be either zero-one or integer. The branch and bound algorithm can be performed to solve ZP or IP. The cutting plane algorithm is another way to solve ZP.

(d) Dynamic Programming (DP)

The DP is used in sequential multiple stage problems which can be expressed by the following figure.



x_k : the k_{th} stage vector (the output of $k+1$ state)

θ_i : the decision variable vector at stage i

f_j : the return function of stage j

The objection function of DP problem can be expressed by

$$f_{n-1}(x_n) = \max_{\theta_{n-1}} \dots \max_{\theta_1} (g_{n-1}(x_n;\theta_{n-1}) + \dots + g_1(x_2;\theta_1))$$

or

$$f_n(x_{n+1}) = \max_{\theta_n} (g_n(x_{n+1};\theta_n) + f_{n-1}(x_n))$$

where g_i is the profit function at stage i

2. Multiple Objective Decision Making (MODM)

Hwang & Masud [15] showed the taxonomy (see Figure 3.18) of MODM in 1978. Typical MODM techniques include Multiple Objective Linear Programming (MOLP), Goal Programming (GP) and Interactive Sequential Goal Programming (ISGP). The general form of the MODM problem can be expressed by

$$\text{Max/Min} \quad \left[f_1(\underline{x}),\ f_2(\underline{x}), \dots,\ f_k(\underline{x}) \right]$$

$$\text{S.T} \quad \underline{x} \in X$$

$$\text{where} \quad X = \left\{ \underline{x} \mid g_i(\underline{x}) \ \{ \geq, =, \leq \} \ 0,\ i = 1,2,\dots,m \right\}$$

3. Mutiple Attribute Decision Making (MADM)

Hwang & Yoon [16] showed the taxonomy (see Figure 3.19) of MADM in 1981. Typical MADM techniques include TOPSIS, and Hierarchical **Additive Weighting method (HAW).**

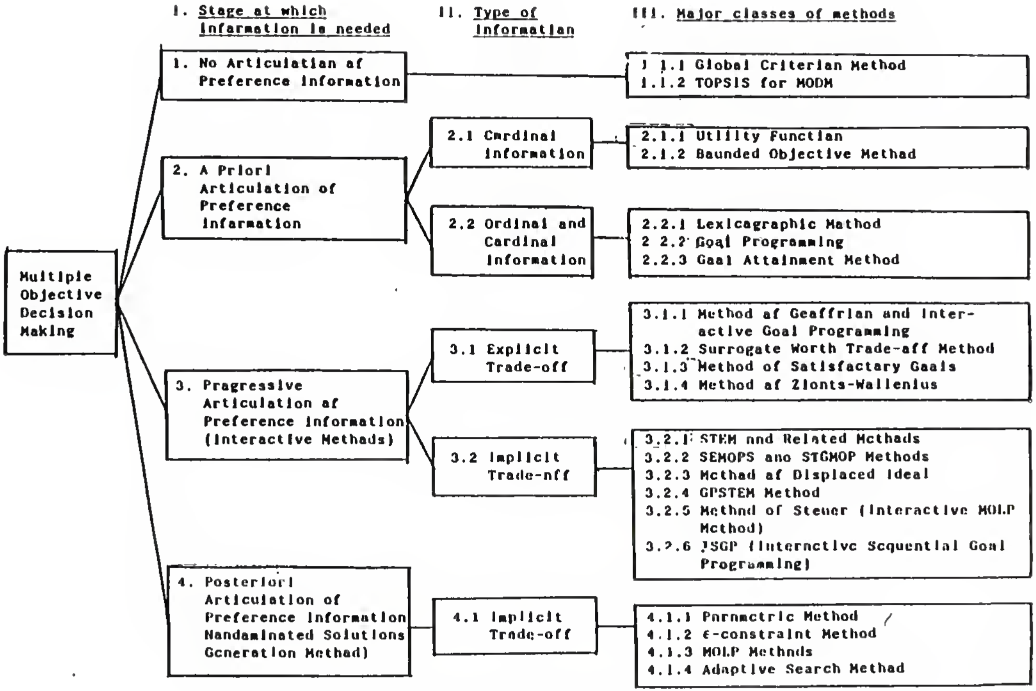


Figure 3.18 The taxonomy of MODM methods [15]

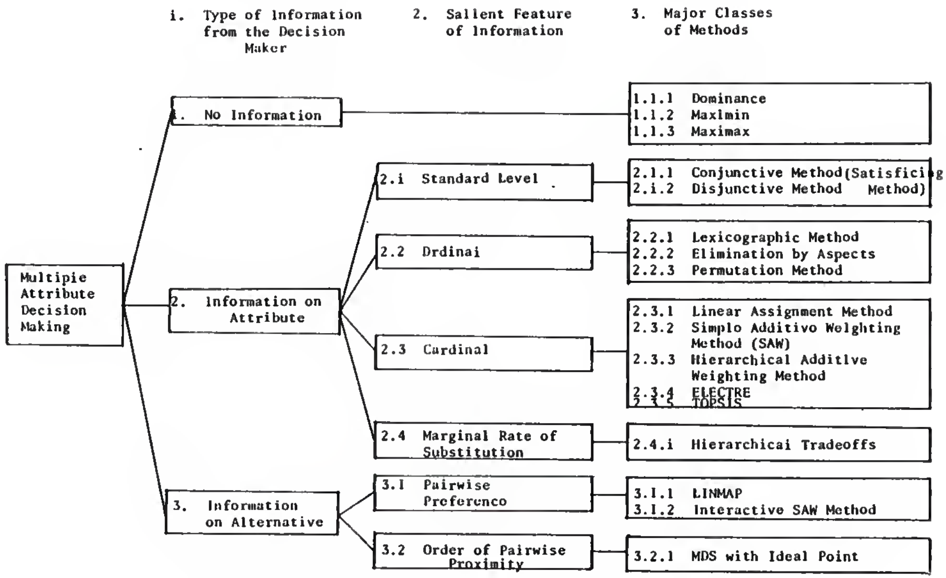


Figure 3.19 The taxonomy of MADM methods [16]

3.5.3 MP examples

Problem:

Three projects in 5 years horizon are considered. The characteristics of the projects and the resources are **shown in the following table.**

	Cash Flow (in thousand)					manpower
	1	2	3	4	5	per year
Proj. 1	-12	3	4	5	6	3
Proj. 2	—	-16	7	8	9	4
Proj. 3	-20	10	-11	18	—	5
Budget	16	6	3	—	—	
Borrowing*	B_1	B_2	B_3	B_4	B_5	rate r_B
Lending*	L_1	L_2	L_3	L_4	L_5	rate r_L

hurdle rate : r

Constraint 1: maximum man power per year is 10
2: borrowing should ≤ 6000 per year

* short-term borrowing and lending on 1 year basis

The decision maker must decide the selection of projects and the borrowing/lending policies.

1. A goal programming model

Suppose the decision maker set two goals for this problem:

(1) the NPV in the 5 year horizon should exceed 10,000

(2) the liquidity objectives require that the NPV should exceed 4000 at end of year 1 and 5000 at end of year 2.

The GP model can be expressed as follows:

$$\text{Min } Z = [\ P1: (\ d_1^-\ ),\ P2: (\ d_2^- + d_3^-\ )$$

Goal 1: (overall NPV)

$$[-12X_1 - 20X_3 + 16 + B_1(1+r_B) - L_1(1+r_L)](1+r)^{-1}$$
$$+ [3X_1 - 16X_2 + 10X_3 + 6 + B_2(1+r_B) - L_2(1+r_L)](1+r)^{-2}$$
$$+ [4X_1 + 7X_2 - 11X_3 + 3 + B_3(1+r_B) - L_3(1+r_L)](1+r)^{-3}$$
$$+ [5X_1 + 8X_2 + 18X_3 + B_4(1+r_B) - L_4(1+r_L)](1+r)^{-4}$$
$$+ [6X_1 - 9X_2 + B_5(1+r_B) - L_5(1+r_L)](1+r)^{-5}$$
$$+ d_1^- - d_1^+ = 10$$

Goal 2: (liquidity for year 1 and 2)

$$-12X_1 - 20X_3 + 16 + B_1(1+r_B) - L_1(1+r_L) + d_2^- - d_2^+ = 4$$

$$[-12X_1 - 20X_3 + 16 + B_1(1+r_B) - L_1(1+r_L)](1+r)$$
$$+ [3X_1 - 16X_2 + 10X_3 + 6 + B_2(1+r_B) - L_2(1+r_L)] + d_3^- - d_4^+ = 5$$

Constraints:

$3X_1 + 4X_2 + 5X_3 \le 10$ (manpower constraint)

$B_i \le 6$ for $i = 1,2,3,4,5$ (borrowing limit)

$0 \le X_j \le 1$ for $j = 1,2,3$

$L_i,\ B_i \ge 0$ for $i = 1,2,3,4,5$

2. A mutiple objective linear pogramming (MOLP) model

For the same problem as in 1 (the GP example), we can also express the problem by a different approach, the MOLP

model. The MOLP model can be expressed by

OBJ:

$$
\begin{aligned}
\text{Max} \quad f_1 = {} & [-12X_1 - 20X_3 + 16 + B_1(1+r_B) - L_1(1+r_L)](1+r)^{-1} \\
& + [3X_1 - 16X_2 + 10X_3 + 6 + B_2(1+r_B) - L_2(1+r_L)](1+r)^{-2} \\
& + [4X_1 + 7X_2 - 11X_3 + 3 + B_3(1+r_B) - L_3(1+r_L)](1+r)^{-3} \\
& + [5X_1 + 8X_2 + 18X_3 + B_4(1+r_B) - L_4(1+r_L)](1+r)^{-4} \\
& + [6X_1 - 9X_2 + B_5(1+r_B) - L_5(1+r_L)](1+r)^{-5}
\end{aligned}
$$

$$
\text{Max} \quad f_2 = -12X_1 - 20X_3 + 16 + B_1(1+r_B) - L_1(1+r_L)
$$

$$
\begin{aligned}
\text{Max} \quad f_3 = {} & [-12X_1 - 20X_3 + 16 + B_1(1+r_B) - L_1(1+r_L)](1+r) \\
& + [3X_1 - 16X_2 + 10X_3 + 6 + B_2(1+r_B) - L_2(1+r_L)]
\end{aligned}
$$

S.T

$$3X_1 + 4X_2 + 5X_3 \le 10$$

$$B_i \le 6 \qquad \text{for } i = 1,2,3,4,5$$

$$0 \le X_j \le 1 \qquad \text{for } j = 1,2,3$$

$$L_i,\ B_i \ge 0 \quad \text{for } i = 1,2,3,4,5$$

3. A MADM model

Let us assume that the decision maker would like to consider the attributes of the three projects. The attributes include: the NPV , the level of risk, the level of liquidity, the manpower requirement, the investment and the

project lives. There are 10 risk levels and 10 liquidity levels. The higher number means higher risk or liquidity.

The MADM model of a three project selection can be expressed as follows:

	NPV	risk level	liquidity level	manpower	cost	proj. life
Proj 1	18000	8	3	3	12	5
Proj 2	12000	4	5	4	16	4
Proj 3	10000	3	4	5	20	4

The typical method to solve the model is TOPSIS [16]. We need to assign weights on attributes and follow the algorithms to solve this model.

Chap 4. Inference System Construction — Systematic Classification of Capital Budgeting Knowledge

The objective of this study is to create a prototype of a knowledge-based capital budgeting expert system. So far, we have discussed quite intensively the knowledge base which is the initial work in developing an expert system. However, an expert system is much more complex.

4.1 Basic Structure of An Expert System

An expert system should include five main components in it [17]: an Input/Output (I/O) system, an inference system, a knowledge base, a global data base and a knowledge acquisition facility.

The Input/Output (I/O) system (or user interface) allows the user to communicate with expert system. It provides a means for the user to enter facts about a specific problem using a problem-oriented language. The processor then interprets questions, commands and information from the user.

The inference system (or control system) is the executive that drive the system. The detail of it will be discussed in section 4.2.

The knowledge base contains the knowledge of the domain which may include facts, beliefs and heuristics. The facts

consist of the knowledge which is widely available and generally agreed upon by experts in the field. The heuristics are the experiential, judgemental knowledge, rule of thumb, or rules of good guessing.

The global data base is the working memory for holding the information on a specific problem. This information includes the input provided by the user through I/O system. It also includes all derived information, the intermediate results which may be used for further decision.

The Knowledge Acquisition (KA) facility allows the system to acquire further knowledge about the domain of the expert system, or even automatically from the libraries, data base and so on. To endow the KA facility with learning capability is an idealized **goal**. It may not be possible for a very long time.

There are three basic players involved in building expert systems; the user, the domain expert and the knowledge engineer. The knowledge engineer is the person with a background in artificial intelligence who builds the expert system. He should obtain the expert knowledge (such as by interviewing experts or by surveying) , organize the knowledge, decide the knowledge representation hierarchy and then help programmers write the codes.

4.2 The Inference System

The inference system is the executive that drives the expert system. It contains the control strategies that enable it to act upon the facts in the global data base and knowledge in the knowledge base to solve the stated problems [Hwang, 17].

The main purpose of an expert system is to offer decision makers intelligent advice or intelligent decisions for their problems. The logic flows to link the knowledge and factors in the global data base are called inference procedures. The inference system includes strategies over the inference procedures so that it is able to connect the knowledge base with the global data base in a systematic manner. There are three most common ways to represent knowledge in the expert system, which are : rules, semantic nets and frames.

A rule is a conditional statement including two parts: (1) one or more IF clauses and (2) one or more THEN clauses upon the IF conditions. For example:

IF client's cash-flow level is high
 AND the client's risk level is high
THEN investment in high-growth stock is recommended.

Sementic nets describe a knowledge representation method based

on a network structure. Semantic nets can establish a property inheritance hierarchy in the nets.

The first example is a semantic net to illustrate the simple fact that "All robins are birds" [3].

$$\text{Robin} \xrightarrow{\text{isa}} \text{Bird}$$

Connecting the nodes is a commonly used link called "isa" which means "is a" or "is a kind of." Properties of objects can be represented in the semantic net. For example, to present the property that robin has wings, we may have the following network:

$$\begin{tikzcd} \text{Robin} \arrow[r, "\text{isa}"] & \text{Bird} \arrow[d, "\text{has-part}"] \\ & \text{wings} \end{tikzcd}$$

In the above net, the inheritance feature would allow us to deduce the fact that robin is a bird and has wings.

Frames are a knowledge representation scheme which store all information about an object or event together. The information about each object is stored in a set of attributes associated with a frame. A patient-frame might look like this:

Patient Frame :

Name : Bob Scott
Birthday: Jan 23, 1955
Sex : Male
Height : 6'2"
Weight : 160 lbs
Blood type : B

A frame can be considered a special type of a small portion of the semantic net. Frames can be linked together to form a hierarchical structure.

The inference system is the executive in an expert system. The role it plays can be seen from Figure 4.1.

In this study, the rule-based programming is used to represent knowledge for the capital budgeting expert system. For a rule-based knowledge representation, there are four basic control strategies: forward chaining, backward chaining, and the searching techniques which include depth-first search and breadth-first search [17].

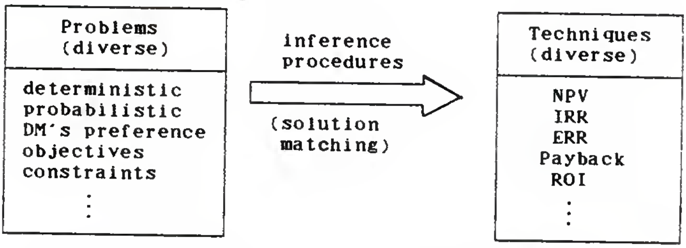


Figure 4.1 The role that inference system plays

Forward chaining is the strategy of working forward from the IF clauses to the THEN clauses of IF-THEN rules. This is the strategy used in our study. Starting with the known facts, the system examines the IF clauses (premises) of the rules to see if any are satisfied. If any are, their THEN clauses (conclusions) are added to the global data base of known facts and the system examine the rules again. Let us explain the strategy with the following example [17]:

Rule 1 : If A, Then B
Rule 2 : If C, Then D
Rule 3 : If E, Then F
Rule 4 : If B and D, Then G
Rule 5 : If B and F, Then H
Rule 6 : If D and F, Then I

Factors in global data bse : A, E
A matches rule 1, therefore B
E matches rule 3, therefore F
B and F match rule 5, therefore H

By the forward chain rules, H is the decision made

Backward chaining is the strategy of working backward from the THEN clauses of a rule to the IF clauses. Here, the system **first starts with a possible goal statement and tries to verify** that it is true or not. Let us see how it works with the same example as shown by the forward chaining.

Rule 1 to Rule 6: (the same as in forward chaining example)
Factors in global data base : A, E
To establish H $\longrightarrow$ need B and F

To establish B ⟶ need A (yes, we have A)

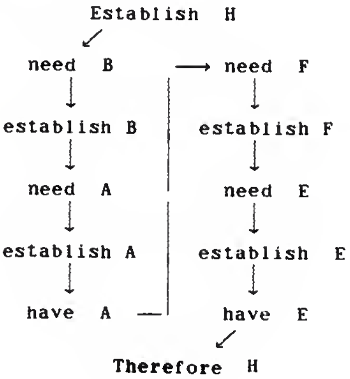
To establish F ⟶ need E (yes, we have E)

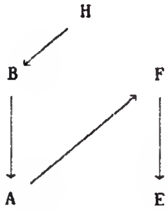
Therefore, H should be established

By backward chaining, H is the decision can be made.

The searching techniques normally refer to the search through a tree or networks of nodes where the nodes represent possible states toward a solution. The search techniques used most frequently by expert systems are depth-first search and breadth-first search [17].

The depth-first search works down a single branch of the tree until it finds a goal node where it can stop or a leaf node where it must back up to a parent node and try a different branch. Let us explain the concept by the example shown in the backward chaining.





The breadth-first search checks all nodes at one level of tree before continuing to the next level in its search for a goal node. The concept can be expressed by using the example of the backward chaining.

Establish H

need B → need F

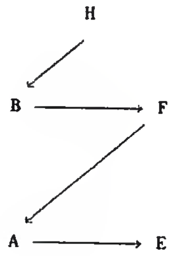
establish B → establish F

need A → need E

establish A → establish E

have A → have E

Therefore H



Ow and Smith [32] declared that the hierarchical organization of **knowledge** sources is an important design principle for knowledge-based expert system. To build the hierarchy of the knowledge representation in the inference system, there are three main tasks should be accomplished: (i) the classification of problems according to the facts behind them, (2) the definition of rules and (3) the hierarchy of the knowledge sources.

4.3 Classification of Capital Budgeting Problems

Capital budgeting problems can be classified into four main categories: (1) project evaluation problems, (2) project ranking problems, (3) portfolio selection problems and (4) mathematical programming problems.

Project evaluation problems refer to the analysis of a single project on its gain/loss, profitability, rate of return and so on. Typical methods in this category include payback period, return on investment, net present value, internal rate of return, external rate of return, risk analysis, certainty-equivalent, decision trees, simulation, Savage principle and so forth.

Project ranking problems refer to the prioritization of projects. Most of the methods used in this category are the same as in the project evaluation category except that their approaches to solutions are quite different. In addition, the following methods are used in this category: expected value and variance principle, aspiration level, most probable future and the MADM techniques.

Portfolio selection problems relate to the investment of fund in efficient portfolios which can maximize the investor's objectives (normally, the expected return) at the investor's level of risk preference. There are variety of theories in

portfolio selection. In this study, we collected only some basic and typical methods. These methods include: Markowitz model, capital asset pricing model, security market line, capital market line and arbitrage pricing model.

Mathematical programming techniques are powerful tools to solve complex problems, such as capital budgeting planning problems or resource allocation problems. The real world capital budgeting problems are always combined with objectives and constraints. Objectives may include the maximization of net profit, the minimization of risk and the minimization of resource usage. Typical constraints may include capital rationing (budget limitation), project interactions, liquidity requirement, positive return necessity, resource limitation and certain boundary constraints. In this study, we classify the mathematical programming problems into three main categories: multiple attribute decision making problems, multiple objective decision making problems and single objective decision making problems.

4.4 Facts of Problems

Prior to the construction of the inference system, we should at least accomplish (1) the collection and classification of facts, (2) the organized hierarchy of knowledge sources and (3) the systematic hierarchy of the inference procedures.

Facts of problems should be defined clearly by specific factors which represent the situations underlying each problem. In capital budgeting problems, the facts are summarized into two groups: the types of condition and the kinds of factors.

4.4.1 Types of Conditions

The problems can be classified according to three categories: deterministic, under risk and under uncertainty.

(1) The deterministic problems

It means that the outcomes of projects are known for sure. Outcomes may include the cash inflows/outflows, useful lives, discount rates, reinvestment retes and so on.

(2) The problems under risk

As we have mentioned in section 4.2, the term 'risk' refers to the situation that we know the number of possible outcomes, the value of each outcome and the probability distribution of each outcome. In this category, we can only get the expected results.

(3) The problems under uncertainty

As we have mentioned in section 4.3, the term 'uncertainty' we defined in this study refers to the situation that we know the number of possible outcomes and the value of each outcome but

we do not know the probability of occurrences of each outcome. In this category, the decision maker can not expect to get an optimum solution from the expert system. Since a lot of guesswork is assumed, the results from the expert system often are more recommendatory than decisive.

4.4.2 Kinds of Factors

The factors are collected from two sources: the facts of problems and the characteristics and the assumptions underlying each solution technique. We would like to summarize the factors in accordance with the four main problem categories.

(1) Factors in project evaluation problems

Factors in this category include

(a). the importance of time value

(b). the discount rate can be determined or not

(c). the reinvestment rates are known or not

(d). prefer liquidity or profitability

(e). compare with hurdle rate or with net cash flow

(f). outcomes are numerous or limited

(g). Are outcomes conditioned on previous ones

(h). DM's understanding of the project is strong or weak

(i). constant or variable risk over project life

(2) Factors in project ranking problems

In addition to the factors of project evaluation problems, there are more factors should be considered in this category.

(a) - (h). **The same factors as in (i)**

(i). DM's utility preference is known or not

(j). DM's degree of optimism

(k). Single attribute or multiple attribute be considered

(3) Portfolio selection Model

Factors in this category can be summarized as follows:

(a). An evaluation problem or a selection problem

(b). DM's preference of risk is known or not

(c). Probability distribution of securities returns are known or not

(d)..Single or multiple index to be used to measure the market return

(e). To measure the portfolio by market risk only or by overall risk

(4) Factors of mathematical programming problems

The factors in this category are summarized according to the characteristics of individual technique.

(a). Measured by objective or by criteria

(b). Single objective or multiple objective

(c). DM's understanding is strong or weak

(d). Whether it is a sequential, multi-stage problem

(e). Linear or nonlinear equations

(f). Interactive or non-interactive

4.5 Inference System Specifications

There are three ways to represent knowledge in expert system: rules, semantic nets and frames. In this study, forward chaining rules are used to construct the inference procedures. One of our goals is to build the systematic hierarchy of inference flows. First, we need to define the rules connecting problems and knowledge. The rules were generated according to individual method and its associated factors. The following tables (Table 4.1 through Table 4.4) show the detail specifications of the rule-based inference programming by the categories of problems.

Table 4.1 Inference rules — project evaluation problems

IF clauses	THEN
1. IF outcomes are deterministic [a] AND time value is important [b] AND hurdle rate is known [c] AND user like to analyze by rate [d] AND reinvestment rate is known [e]	IRR
2. IF [a] AND [b] AND [c] AND [d] AND not [e]	ERR

(Continue to next page)

Table 4.1 Inference rules — project evaluation problems (Contd.

IF clauses	THEN
3. IF outcomes are deterministic [a] AND time value is important [b] AND hurdle rate is known [c] AND **need to analyze the net cash flow** [f]	PV, FV or AW
4. IF [a] AND [b] AND [c] AND [f] AND **want to analyze the profitability** [g]	PI
5. IF [a] AND not [b] AND **want to analyze the liquidity**	ROI
6. IF [a] AND not [b] AND **[g]**	PAYBACK
7. IF not [a] AND probability distribution is known [i] AND outcomes are numerous [j]	SIMULATION
8. IF not [a] AND [i] AND not [j] AND outcomes are conditioned [k]	DECISION TREES
9. IF not [a] AND [i] AND not [j] AND not [k]] AND constant risk over project life	RISK ANALYSIS or CERTAINTY EQUIVALENT
10. IF not [a] AND not [i] AND DM's undertanding is strong	HURWICZ
11. IF not [a] AND not [i] AND DM's understanding is weak	LAPLACE

Table 4.2 shows the inference rules of project ranking

problems. Most of the rules are similar to the rules in project evaluation problem except that incremental cash flow approaches and some remedial steps are imposed to recover the disparities resulting from multiple project comparisons.

Table 4.2 Inference rules — project ranking problems

IF clauses	THEN
rules 1, 2, 3, 4, 5, 6 of project evaluation problems	
7. IF not deterministic [a] AND not (Probability distribution is known [i]) AND not (measured by single attributes [l])	MADM
8. IF not [a] AND not [i] AND [l] AND DM's understanding is weak [m]	LAPLACE
9. IF not [a] AND not [i] AND [l] AND not [m] AND index of optimism can be defined [n]	HURWICZ
10. IF not [a] AND not [i] AND [l] AND not [m] AND not [n] AND DM's is very optimistic	MAXIMAX / MINIMIN
11. IF not [a] AND not [i] AND [l] AND not [m] AND not [n] AND DM's is very pessimistic	SAVAGE or MAXIMIN / MINIMAX
12. IF not [a] AND [l] AND utility preference is known [o]	E-V PRINCIPLE

(Continue to next page)

Table 4.2 Inference rules — project ranking problems (Contd.)

IF clauses	THEN
13. IF not(deterministic [a]) AND probability distribution is known [i] AND not (utility preference is known [o]) AND outcomes are limited [p] AND DM is very optimistic (q)	MOST PROBABLE
14. IF not [a] AND [i] AND not [o] AND outcomes are numerous	ASPIRATION LEVEL
15. IF not [a] AND [i] AND not [o] AND [p] AND not [q] AND variable risk over project life	RISK ADJUSTED
16. IF not [a] AND [i] AND not [o] AND [p] AND not [q] AND constant risk over project life	RISK ANALYSIS or CERTAINTY EQUIVALENT

Table 4.3 shows the inference rules regarding portfolio theory. The classification in this study is still primitive since the field in portfolio theory is so diversified and so broad. However, the basic and typical methods are included in our category.

Table 4.4 shows the inference rules of mathematical programming prpoblems. Although there are numerous methods in this category, we only pick out the typical methods which are proper to be used in our system.

Table 4.3 Inference rules — portfolio selection problem

IF clauses	THEN
1. IF for evaluation purpose	MARKOWITZ
2. IF for ranking or selection [a] AND not (DM's risk preference is known [b]) AND probability distribution is known [c]	L-C-L CRITERION
3. IF [a] AND measure by multiple market index	APM
4. IF [a] AND [b] AND measure by single index [d] AND measure by market risk only	SML
5. IF [a] AND [b] AND measure by single index [d] AND measure by total risk	CML

Table 4.4 Inference rules — Mathematical programming

IF clauses	THEN
1. IF measure by multiple attributes	MADM
2. IF single objective [a] AND all linear equations [b]	LP
3. IF single objective [a] AND integer variables imposed	IP or ZP
4. IF [a] AND not (all linear equations [b])	NLP
5. IF [a] AND integer variables imposed	IP or ZP

(Continue to next page)

Table 4.4 Inference rules — Mathematical programming (Contd.)

IF clauses	THEN
6. IF single objective [a] AND a sequential, multi-stage problem	DP
7. IF multiple objective [c] AND DM's knowledge is strong [d]	GP
8. IF [c] AND not [d] AND interactive with DM	ISGP
9. IF [c] AND not [d] AND linear equations	MOLP

4.6 Inference System Modules

According to the rules defined for the inference procedures, We can analyze the whole flow and then build the hierarchy of the inference system by modules. Figure 4.2 through Figure 4.5 show the systematic hierarchy of the inference system modules.

Figure 4.2 : The project evaluation module

Figure 4.3 : The project ranking module

Figure 4.4 : The portfolio selection module

Figure 4.5 : The mathematical programming module

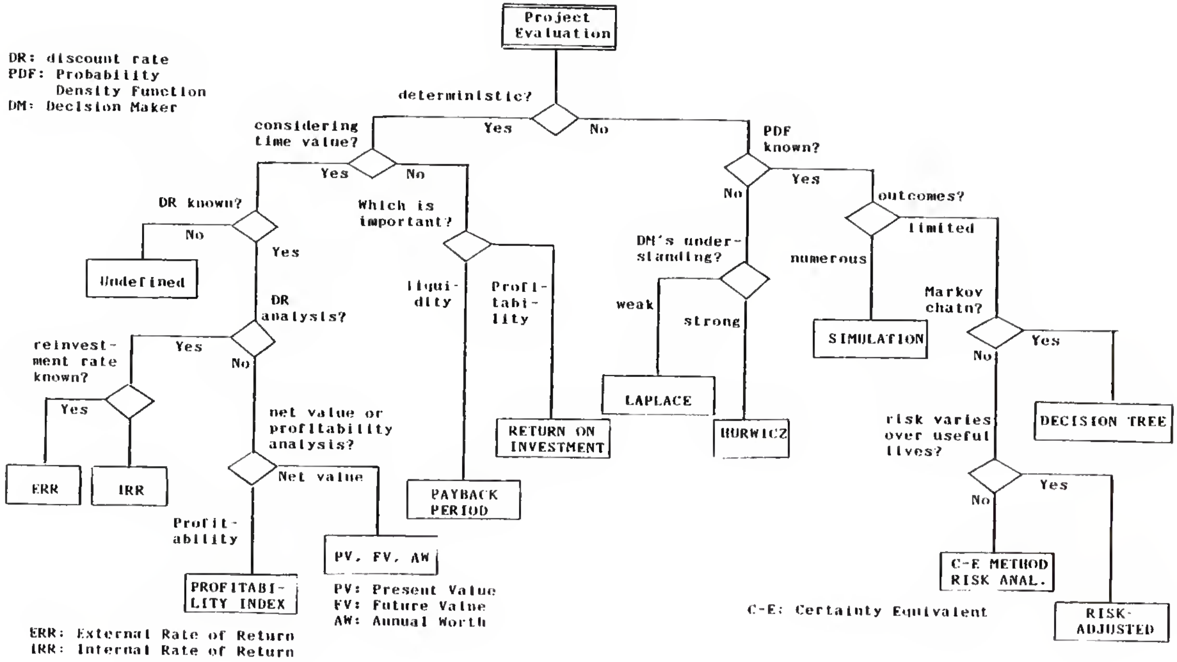


FIGURE 4.2 PROJECT EVALUATION MODULE

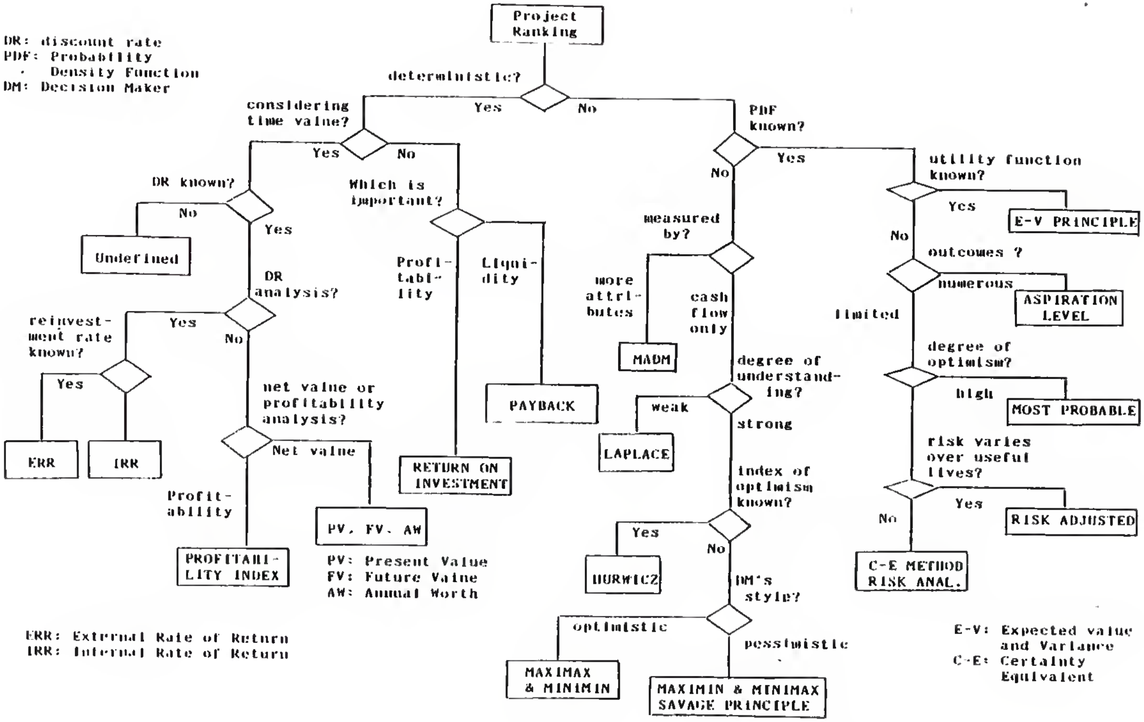
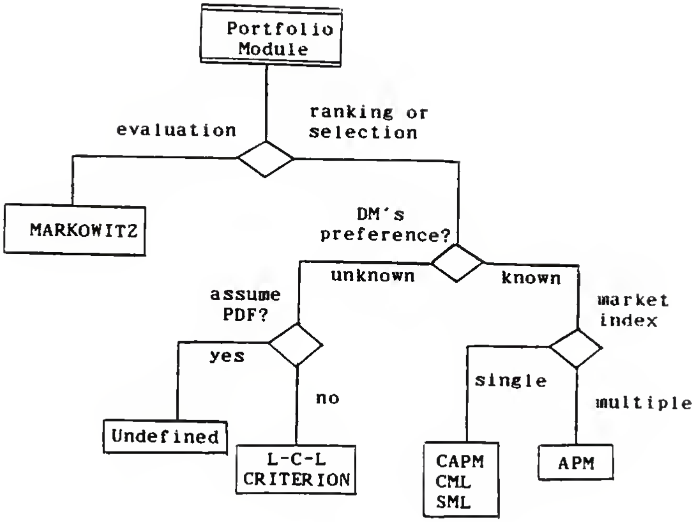
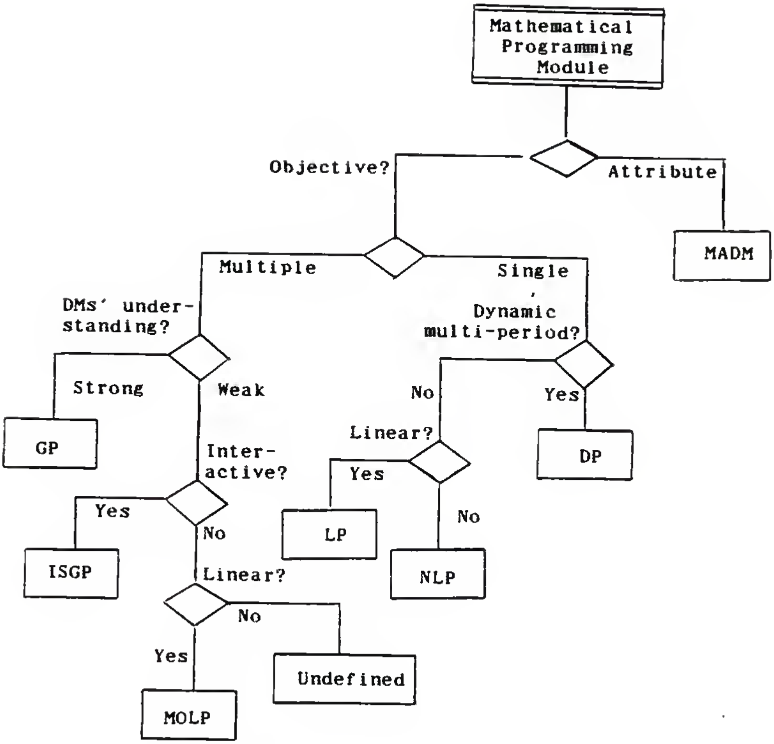


FIGURE 4.1 PROJECT RANKING MODULE



L-C-L: Lower-Confidence-Limit
CAPM : Capital Asset Pricing Model
APM : Arbitrage Pricing Model
CML : Capital Market Line
SML : Security Market Line

FIGURE 4.4 PORTFOLIO SELECTION MODULE



MADM: Multiple Attribute Decision Making
GP: Goal Programming
ISGP: Interactive Sequential Goal Programming
MOLP: Multi-Objective Linear Program
LP: Linear Programming
NLP: NonLinear Programming
DP: Dynamic Programming

FIGURE 4.5 MATHEMATICAL PROGRAMMING MODULE

Chap 5. Summary and Conclusion

There are five main components in an expert system: I/O system, knowledge acquisition facility, global data base, knowledge base and inference system. In the thesis, we focus on the construction of knowledge base and the inference system of the capital budgeting expert system.

We have collected and discussed in detail the techniques for solving capital budgeting problems. With the understanding of capital budgeting knowledge, we then systematically classified these techniques into three main categories: project evaluation and ranking (under certainty, under risk, and under uncertainty), portfolio selection and mathematical programming. Figure 5.1 shows the taxonomy of capital budgeting techniques in the thesis. These tehniques are the fundamental works in constructing the knowledge base of an expert system.

By analyzing the facts of capital budgeting problems and the assumptions and conditions underlying the techniques for solving them, we have defined the rules for knowledge representation. These rules are incorporated into four main modules: project selection, project ranking, portfolio selection and mathematical programming. The logic flows of inference procedures can be seen clearly after the systematic hierarchies for each module were built.

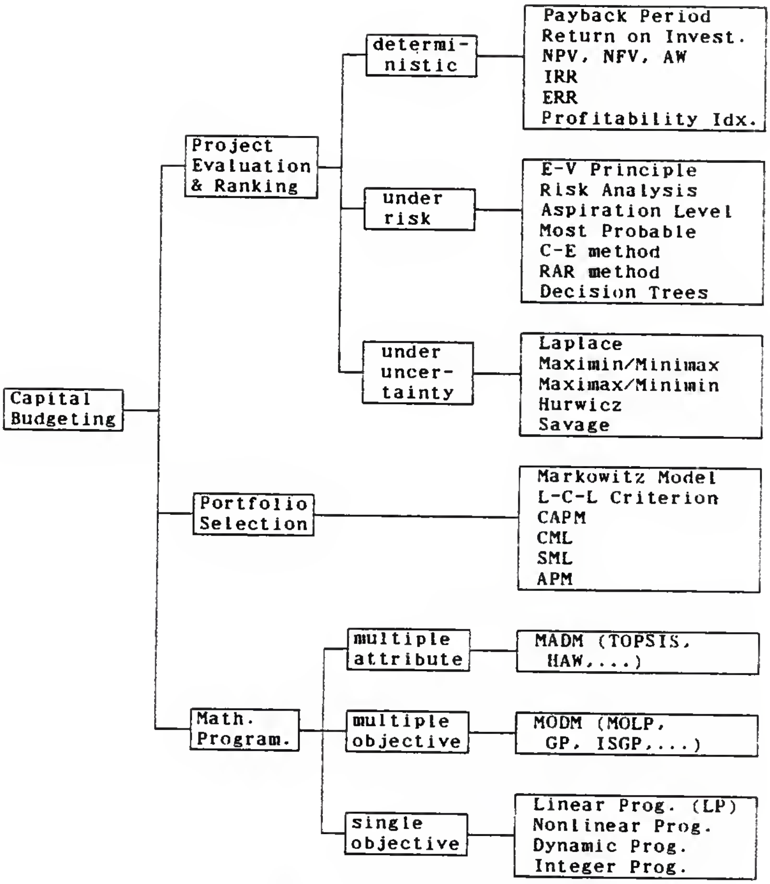


Figure 5.1 Taxonomy of capital budgeting methods

Khan's survey [22] on capital budgeting practices in U. S. big cities revealed that people were lacking understanding in capital budgeting techniques and only a few simple techniques were used. No company can afford losses due to poor decisions, especially capital budgeting decisions which always relate to a lot of money. Thus, the necessity of an expert system supporting capital budgeting decisions is foreseen in the near future.

We have constructed the fundamental works of knowledge base and inference system for a capital budgeting expert system. It is a long, long way to implement a real applicable expert system. In this thesis, we have proposed and established the prototype of a capital budgeting expert system which will pave the way for its further development.

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TOWARD A KNOWLEDGE-BASED EXPERT SYSTEM

OF CAPITAL BUDGETING DECISION MAKING

by

JAN-SUAN LIN

B.S., National Chiao-Tung University, Taiwan, 1979

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ABSTRACT

Capital budgeting is the decision making area on the use of fund in long-term projects. Typical examples may include the investments on buildings, equipments, facilities, stock markets and so forth. An expert system is a computer program which solves problems in much the same manner as human experts.

The purpose of the thesis is trying to build the prototype of a knowledge-based expert system which may comprise variety of techniques and is able to solve diverse capital budgeting problems. Two main constructions were made by the study, the knowledge base and the inference system.

In knowledge base construction, the capital budgeting techniques were classified into three categories: project evaluation and ranking , portfolio selection and mathematical programming. In inference system construction, there are four main modules: project evaluation, project ranking, portfolio selection and mathematical programming. The systematic hierarchies of inference flows were build according to the facts of problems and the conditions and assumptions underlying each technique.